ON ADDING INDEPENDENT STOCHASTIC PROCESSES

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Let x be a stochastic process on an interval T, P_x be the probability measure it induces on the space Ω of sample functions on T, and M be the set of functions f in Ω such that P_{x+f} is absolutely continuous with respect to P_x (written $P_{x+f} \prec P_x$). The functions in M are "at least as smooth as" the sample functions of x in the sense that any smoothness property possessed by almost all of the sample functions must also be possessed by each function in M. Similarly, if y is a process independent of x such that $P_{x+y} \prec P_x$, then almost all the sample functions of y must possess any smoothness property possessed by almost all the sample functions of x. This might lead one to conjecture that $P_{x+y} \prec P_x$ would imply $P_y(M) = 1$ but this fails to hold even in the Gaussian case [1]. What is true in the Gaussian case is that $P_y(M) = 1$ if and only if the measure Q associated with the vector process $(x + y, y)$ is absolutely continuous with respect to the measure P associated with the process (x, y). The theorem of this note generalizes this result to a large class of separable x's.

We assume that x is separable in the sense that there exists a countable subset (t_i) of T such that the closure, with respect to P_x, of the σ-field generated by the $x(t_i)$ contains the σ-field generated by all the $x(t)$. We also assume that the joint distribution of $x(t_1), \cdots, x(t_n)$ is given by a density G_n which is almost everywhere positive in R^n with respect to Lebesgue measure. We shall write x_i, y_i, f_i, and g_i for $x(t_i), y(t_i), f(t_i)$, and $g(t_i)$ and S_n for the σ-field generated by $x_1, \cdots, x_n$. It is easily verified that the function D_n on $\Omega \times \Omega$ defined by

$$D_n(f, g) = G_n(f_1 - g_1, \cdots, f_n - g_n)/G_n(f_1, \cdots, f_n)$$

is, for each fixed g in Ω, a martingale with respect to the fields S_n and the measure P_x. It follows that $D_n(f, g)$ converges almost everywhere (P_x) for each g to a limit $D(f, g)$, that $\int D(f, g)P_x(df) \leqq 1$ and that $\int D(f, g)P_x(df) = 1$ if and only if g is in M, in which case $D(f, g) = (dP_{x+g}/dP_x)(f)$ almost everywhere (P_x).

Theorem. *If x and y are independent processes on T and P and Q are the measures associated with (x, y) and $(x + y, y)$ then $Q \prec P$ if and only if $P_y(M) = 1$. In this case $(dQ/dP)(f, g) = D(f, g)$.*

Proof. Suppose $Q \prec P$. For any $S_n \times S_n$ measurable F

$$\int FD_n\,dP = \iint F(f_1, \cdots, f_n, g_1, \cdots, g_n) \cdot G_n(f_1 - g_1, \cdots, f_n - g_n)\,df_1 \cdots df_n\,P_y\,(dg)$$

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$$= \iint F(f_1 + g_1, \cdots, f_n + g_n, g_1, \cdots, g_n) \cdot G_n(f_1, \cdots, f_n)\, df_1 \cdots df_n\, P_y\,(dg)$$

$$= \int F\, dQ$$

so $D_n = E(dQ/dP \mid S_n \times S_n)$ and hence $D = dQ/dP$. Thus

$$1 = \int D\, dP = \int P_y\,(dg) \left(\int D(f, g) P_x\,(df) \right)$$

so

$$P_y \left(\left[g \,\middle|\, \int D(f, g) P_x\,(df) = 1 \right] \right) = P_y(M) = 1.$$

Conversely, if $P_y(M) = 1$ then $\int D\, dP = \int P_y(dg) \int D(f, g) P_x(df) \geqq P_y(M) = 1$ and this plus the fact that for any positive $S_n \times S_n$ measurable function F, $\int FD\, dP \leqq \liminf_{m\to\infty} \int FD_m\, dP = \int F\, dQ$ implies that $Q < P$ and $D = dQ/dP$.

REFERENCE

[1] PITCHER, T. S. (1963). On the sample functions of processes which can be added to a Gaussian process. *Ann. Math. Statist.* **34** 329–333.