A NOTE ON THE TRUNCATED EXPONENTIAL DISTRIBUTION[1]

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1. Introduction. Deemer and Votaw [1] derived the maximum likelihood estimator of the parameter c in the truncated exponential distribution,

$$(1)\quad \begin{aligned} f(x; c) &= c \exp(-cx)(1 - e^{-cx_0})^{-1} \qquad && 0 < x < x_0 \\ &= 0 && \text{otherwise} \end{aligned}$$

and they give asymptotic confidence limits for the parameter based on the maximum likelihood estimator using large sample theory. In this note we obtain the exact distribution of the minimal sufficient statistic for this family of distributions.

Based on this distribution then, uniformly most powerful (UMP) and uniformly most powerful unbiased (UMPU) tests of hypotheses about the parameter c can be obtained. Uniformly most accurate (UMA) and uniformly most accurate unbiased (UMAU) [2] confidence limits on the parameter c can also be obtained.

2. Results. The joint distribution of a sample of n independent observations in the distribution (1) is given by

$$\begin{aligned} g(x_1, x_2, \cdots, x_n; c) &= c^n[1 - \exp(-cx_0)]^{-n} \exp\left(-c \sum_{i=1}^{n} x_i\right) \\ & \qquad\qquad 0 < x_i < x_0, i = 1, 2, \cdots, n \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

Since this constitutes an exponential family, attention can be restricted to the consideration of the statistic $y = \sum_{i=1}^{n} x_i$. Furthermore, it follows from the theory of exponential families (see for example Lehmann [2]) that there exist uniformly most powerful or uniformly most powerful unbiased tests as well as optimum exact confidence limits for the parameter c. To make these procedures explicit, it is only necessary to obtain the distribution of y, which will now be done by the method of characteristic functions [3].

Since the x_i's are assumed independent, the characteristic function of $\sum_{i=1}^{n} x_i = y$ is given by

$$\phi_y(t) = [\phi_x(t)]^n = b^n c^n (c - it)^{-n}[1 - e^{-x_0(c-it)}]^n,$$

where $b = (1 - e^{-cx_0})^{-1}$.

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Using the inversion formula for the characteristic function the density function of y is found to be

$$g(y;c) = \frac{(bc)^n}{(n-1)!} e^{-cy} \sum_{k=0}^{k_0} (-1)^k \binom{n}{k} (y - kx_0)^{n-1}; \quad k_0 x_0 < y < (k_0+1)x_0,$$

where $k_0 = 0, 1, \cdots, n-1$.

The cumulative distribution function of y evaluated at a is $G_y(a;c) = \int_0^a g(y;c)dy$.

Letting $a = (k_0' + d)x_0$, where k_0' is an integer and $0 \leqq d < 1$, then the cumulative distribution of y is given by

$$G_y[(k_0' + d)\,x_0; c] = [1 - e^{-cx_0}]^{-n} \sum_{k=0}^{k_0} (-1)^k \binom{n}{k} e^{-ckx_0}\, Q\,[2(k_0' - k + d)\,x_0\,c, 2n],$$

where $Q(u, v)$ denotes $\text{Prob}\{\chi^2(v) < u\}$ where $\chi^2(v)$ is a chi-square variable with v degrees of freedom [4].

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