ESCAPE PROBABILITY FOR A HALF LINE

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Let $\{X_n\}$, $n > 0$ be a sequence of independent, identically distributed random variables. Suppose $E|X_1| < \infty$ and $EX_1 > 0$. Then the strong law of large numbers assures us that from any initial point S_0, the random walk

$$S_n = S_0 + X_1 + \cdots + X_n$$

lies in the interval $(-\infty, 0]$ for at most a finite number of values of n with probability one, and thus the non-positive axis is a transient set for this Markov process. Let $M_n = \min(S_1, S_2, \cdots, S_n)$. As M_n is non-increasing, we have that $M = \lim_{n\to\infty} M_n = \inf_{n\geqq 1} M_n$, exists, and as $P(M = -\infty) \leqq P(S_n \leqq 0$ i.o.$) = 0$, we have that M is finite with probability one. Starting at a point x on the non-positive axis let us define the *escape function* $e(x)$ as the probability that a particle, initially at x, will in the first step enter the positive axis and thereafter never return to the non-positive axis. It will be convenient, and in accord with the potential theory for Markov processes, to define $e(x) = 0$ for $x > 0$. More precisely then, we define

$$\begin{aligned} e(x) &= P(M > 0 \mid S_0 = x) & \text{if } x \leqq 0 \\ &= 0 & \text{if } x > 0. \end{aligned}$$

Our principal aim in this note will be to establish the following result.

Theorem 1. *If* $E\,|X_1| < \infty$ *and* $EX_1 > 0$ *and if* Z *is the first positive partial sum starting from* $S_0 = 0$, *then*

(1) $$e(x)/EX_1 = P(Z > -x)/EZ \qquad \text{if } x \leqq 0$$

and

(2) $$\int_{-\infty}^{\infty} e(x)\,dx = EX_1 .$$

Proof. For the sequence $\{S_n\}$ with $S_0 = 0$ let

$$\begin{aligned} W &= \inf\{k > 0: S_k > 0\}, && \text{if for some positive integer } n,\ S_n > 0, \\ &= \infty, && \text{otherwise;} \\ W' &= \inf\{k > 0: S_k \leqq 0\} && \text{if for some positive integer } n,\ S_n \leqq 0, \\ &= \infty, && \text{otherwise.} \end{aligned}$$

On the event $[W < \infty]$ let $Z = S_W$, and on the event $[W' < \infty]$ let $Z' = S_{W'}$. A basic identity in the fluctuation theory for sums S_n([4] Theorem III.6–

Received 15 November 1963; revised 5 February 1964.

III.9) asserts that for λ real and $|t| < 1$ we have

$$(3)\quad \begin{aligned} [1 - E(e^{i\lambda Z}t^{W}; W < \infty)] &= \exp\left[-\sum_{k=1}^{\infty} t^k E[e^{i\lambda S_k}; S_k > 0]k^{-1}\right] \\ 1 - E[e^{i\lambda Z'}t^{W'}; W' < \infty] &= \exp\left[-\sum_{k=1}^{\infty} t^k E[e^{i\lambda S_k}; S_k \leqq 0]k^{-1}\right], \end{aligned}$$

where here and in the following, for an event A and a random variable f we shall let $\int_A f\,dP = E[f; A]$. Thus, for example

$$E(e^{i\lambda S_k}; S_k > 0) = \int_{[S_k(w)>0]} e^{i\lambda S_k(w)}\,dP(w) = \int_{0+}^{\infty} e^{i\lambda x} P(S_k \,\varepsilon\, dx)$$

and

$$E[e^{i\lambda Z'}t^{W'}; W' < \infty] = \int_{[W'(w)<\infty]} e^{i\lambda Z'(w)} t^{W'(w)}\,dP(w)$$

$$= \int_0^{\infty} \sum_{k=1}^{\infty} t^k e^{i\lambda x} P(Z' \,\varepsilon\, dx, W' = k).$$

From (3) we obtain,

$$(4)\quad [1 - E(e^{i\lambda Z}t^{W}; W < \infty)]\,[1 - E(e^{i\lambda Z'}t^{W'}; W' < \infty)] = 1 - tE(e^{i\lambda X_1}).$$

Under the condition we are assuming here, Spitzer ([5] Theorem 3.4) showed that $EZ = EWEX_1$ and $EW < \infty$. If we set $\lambda = 0$ in (4) we obtain the identity,

$$(1 - Et^{W})\,(1 - E(t^{W'}; W' < \infty)) = (1 - t)$$

and by dividing both sides of the above by $(1 - t)$ and then taking the limit as $t \to 1^-$ we obtain the identity, $EWP(W' = \infty) = 1$, and thus $EZP(W' = \infty) = EX_1$. Again from (4) (with $t = 1$) we obtain the identity

$$(5)\quad \frac{[1 - E(e^{i\lambda Z})]}{EZ}\,\frac{[1 - E(e^{i\lambda Z'}; W' < \infty)]}{P(W' = \infty)} = \frac{1 - E(e^{i\lambda X_1})}{EX_1}$$

By Corollary III.10 of [4] we have for λ real and $|t| < 1$ that

$$(6)\quad \begin{aligned} 1 - \sum_{n=0}^{\infty} t^{n+1}\{E(\exp(i\lambda M_n^-)) - E(\exp(i\lambda M_{n+1}^-))\} \\ = (1 - t)\sum_{n=0}^{\infty} t^n E(\exp(i\lambda M_n^-)) \\ = [1 - E(t^{W'}; W' < \infty)]/[1 - E(e^{i\lambda Z'}t^{W'}; W' < \infty)], \end{aligned}$$

where here and in the following, $M_n^- = \min(0, M_n)$, and $M_0 = 0$. Now with probability one, we have, $\lim_{n\to\infty} M_n^- = \min(0, \inf_{n\geqq 1} M_n) = \min(0, M) = M^-$ and $\lim_{n\to\infty} M_n^+ = \lim_{n\to\infty} \max(0, M_n) = \lim_{n\to\infty} (M_n - M_n^-) = M - M^- = M^+$. At $t = 1$, the series on the left hand side of (6) becomes the series

$$1 - \sum_{n=0}^{\infty} [E(\exp(i\lambda M_n^-)) - E(\exp(i\lambda M_{n+1}^-))]$$
$$= \lim_{n\to\infty} E(\exp(i\lambda M_n^-)) = E(\exp(i\lambda M^-))$$

while at $t = 1$ the expression on the right in (6) becomes $P(W' = \infty)[1 - E(e^{i\lambda Z'}; W' < \infty)]^{-1}$. Thus we have the identity,

$$(7) \qquad E(\exp(i\lambda M^-)) = P(W' = \infty)\,[1 - E(e^{i\lambda Z'}; W' < \infty)]^{-1}.$$

On the other hand,

$$(8) \qquad E(\exp(i\lambda M_n^-)) + E(\exp(i\lambda M_n^+)) = E(\exp(i\lambda M_n)) + 1.$$

It is a well known fact (see [6], Section 2) that

$$(9) \qquad E(\exp(i\lambda M_n)) = E(\exp(i\lambda M_{n-1}^-))E(\exp(i\lambda X_1)).$$

From (8) and (9) we obtain (by letting $n \to \infty$) the following relation,

$$(10) \qquad [1 - E(\exp(i\lambda X_1))]E(\exp(i\lambda M^-)) = 1 - E(\exp(i\lambda M^+)).$$

If we substitute (7) into (10) we obtain

$$1 - E(\exp(i\lambda M^+)) = [1 - E(e^{i\lambda X_1})]P(W' = \infty)[1 - E(e^{i\lambda Z'};\ W' < \infty)]^{-1},$$

and from (5) we then have,

$$[1 - E(e^{i\lambda M^+})]/EX_1 = [1 - E(e^{i\lambda Z})]/EZ.$$

The uniqueness theorem for characteristic functions then gives us that

$$(11) \qquad P(M^+ > x)/EX_1 = P(Z > x)/EZ, \qquad \text{for all } x \geqq 0.$$

Finally, from the definition of $e(x)$ for $x \leqq 0$ we have $e(x) = P(M > -x) = P(M^+ > -x)$, which establishes (1). (2) follows at once from (1).

As a consequence of the proof we have the following

Corollary 2. *If* $E|X_1| < \infty$ *and* $EX_1 > 0$ *then*

$$(12) \qquad P(M^+ > x) = EX_1 P(Z > x)/EZ$$

and thus

$$(13) \qquad EM^+ = EX_1\,.$$

The relation (13) was first found by Dwass [1] for the special case when the X_1 are integer valued. Relation (12) then offers an explanation of this phenomena.

Let A be a Borel set on the positive axis, and let $H(x; A)$ be the probability that a particle initially at x first hits the positive axis at a point in A. In precise terms,

$$H(x; A) = \delta_x(A), \qquad x > 0$$
$$= \sum_{n=1}^{\infty} P(S_n\,\varepsilon\,A,\ S_r \leqq 0,\ 1 \leqq r < n \mid S_0 = x), \qquad x \leqq 0.$$

We then have

THEOREM 3. *Assume* $E|X_1| < \infty$ *and* $EX_1 > 0$. *Then if* X_1 *has a non lattice distribution,*

$$\lim_{x\to-\infty} H(x; A) = \int_A \frac{e(-x)}{EX_1}\, dx \tag{14}$$

while if X_1 *has a lattice distribution, we may with no loss in generality, assume that the smallest additive subgroup of reals which contain the points of increase of the distribution of* X_1 *to be the group of all integers. In this case we have for any positive integer* k,

$$\lim_{n\to-\infty} H(n; \{k+1\}) = e(-k)/EX_1 . \tag{15}$$

Before giving the proof of the above assertions, let us offer the following heuristic explanation of say (15) which was suggested by T. E. Harris. Consider the path of a particle which starts at $-\infty$ and first enters the positive axis at the point k, *backwards*. This is just the path of a particle which starting from $-k$ escapes the negative axis on its first transition and thereafter drifts to ∞.

PROOF. Let $\{Z_n\}$ be the positive ladder random variables for the sums S_n (see [4], Section III). We then have that $\{Z_k\}$ are independent, positive, random variables each with the distribution of Z. The sums, $Z_1, Z_1 + Z_2, \cdots, Z_1 + \cdots + Z_n, \cdots$, thus constitute a positive renewal process. Let V_t be the "excess" random variable for this renewal process, i.e., V_t is the amount over a barrier at t by which the sums $Z_1, Z_1 + Z_2, \cdots$, first exceed that barrier. A little reflection shows for any positive Borel set A we have, for $x \leqq 0$ that, $H(x; A) = P(V_{-x} \in A)$.

Consider the case when X_1 has a non lattice distribution. Then Z also has a non lattice distribution. A well known theorem in renewal theory (see [3]; I) then asserts that

$$\lim_{t\to\infty} P(V_t \in A) = \int_A \frac{P(Z_1 > x)}{EZ_1}\, dx,$$

from which we obtain (14) at once by Theorem 1.

When X_1 has a lattice distribution as described above, we then have that the sums $Z_1, Z_1 + Z_2, \cdots$, constitute an aperiodic, discrete, renewal sequence. If $u_n = \sum_{k=1}^{n} P(Z_1 + \cdots + Z_k = n)$, then the fundamental theorem of recurrent events asserts that $\lim_{n\to\infty} u_n = (EZ_1)^{-1}$. (For details on recurrent events see Chapter 13 of [2].) It is readily seen that

$$P(V_n \leqq r) = \sum_{k=0}^{r-1} u_{n+r-k} P(Z_1 > k)$$

and thus

$$\lim_{n\to\infty} P(V_n = r) = P(Z_1 \geqq r)/EZ_1 .$$

Use of Theorem 1 now establishes (15).

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