be the theory of weakly stationary random fields on a group where the random variables of the field are real or complex valued and it is the indexing parameter set that is a topological group. Although this is not a case of probability "on" a general algebraic structure, the latter enters in an essential way (via stationarity) in the study of the process.

The reader of this review will have guessed by now that the book's mathematical prerequisites go considerably beyond a knowledge of standard measure theory and of advanced probability theory, say at the level of Loève's book. No secret is made of this fact in the Introduction, which expects that the student who wishes to work through the text should have familiarity with the elements of functional analysis (as in the well known book by Hille and Phillips) and topological algebra (as in Neumark's *Normed Rings*). I think that advanced graduate students intending to specialize in abstract probability theory will find this book a very useful text for seminars.

Almost all the topics treated in the book are areas in which active research is still being done, and the difficulties in presenting a logical development of the principal results in a fairly slim volume are many. I, for one, am glad that Professor Grenander has courageously undertaken the task.

Yu. V. Linnik, *Décomposition des lois de probabilités* (translated from the Russian by M. L. Gruel) (Monographies internationales de mathématiques modernes), Gauthier-Villars, Paris, 1962. Fr 55.—vi + 294 pp.

Yu. V. Linnik, *Decomposition of probability distributions* (edited by S. J. Taylor), Oliver and Boyd Ltd., London, 1964. £4/4/0.—239 pp.

Review by E. Lukacs

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One of the most important topics in classical probability theory is the addition of independent random variables, studied since the first quarter of the 18th century. The converse problem, the decomposition of a given random variable into independent summands, is of recent origin and was started by investigations of P. Lévy, A. Ya. Khintchine, H. Cramér, and D. A. Raikov during the late thirties of the present century. The last author raised a number of interesting questions [1] concerning the family of infinitely divisible laws which have no indecomposable components. No appreciable progress was made towards the solution of these problems until the studies of Yu. V. Linnik which were published in the Russian probability journal during the years 1957–1959. In the present monograph, the author gives a systematic and connected account of his investigations.

The theory of decomposition of random variables, often called the arithmetic of distribution functions, deals with problems on the borderline between probability theory and classical analysis. In order to make the presentation self-contained, the author presents in the first chapter a number of important results from the theory of functions of real and complex variables. Results which are not easily found in standard textbooks are given with detailed proofs (a lemma due to Vinogradov and the Paley-Wiener theorem belong in this group), while more easily accessible results, for example theorems concerning entire functions, are stated without proof. Chapters 2 to 6 give a survey of the theory of characteristic functions. In Chapter 2 a number of basic theorems are presented, Khintchine's functional as well as Lévy's concentration function are introduced. A very elegant new proof of Bochner's theorem on the representation of positive definite functions is given. The relation between characteristic functions and moments and some properties of analytic characteristic functions are treated in Chapter 3. Chapter 4 deals with the very important factorization theorem due to D. A. Raikov and with related topics. Raikov's theorem states that the factors of analytic characteristic functions are necessarily also analytic characteristic functions. The idea of α-factorizations is also introduced in this chapter and a basic theorem is proven. The α-factorizations deal with a situation where s characteristic functions $\varphi_j(t)$ and s positive constants α_j $(j = 1, 2, \cdots, s)$ are given and the validity of a relation

$$\prod_{j=1}^{s} [\varphi_j(t)]^{\alpha_j} = \varphi(t) \tag{1}$$

is assumed. Equation (1) may be valid in a finite t-interval or alternatively for a sequence of t values converging to 0. Suitable assumptions concerning $\varphi(t)$ are made [f.i. regularity] and similar analytical properties of the $\varphi_j(t)$ are established. The α-factorizations demonstrate that the arithmetic of distribution functions occupies a position between probability theory and the theory of functions. While these problems have a probabilistic motivation and origin, they cannot be reformulated (for α_j-irrational) in terms of random variables. Indecomposable distributions and Khintchine's general factorization theorems are presented in Chapter 5. Chapter 6 deals with factorization problems of infinitely divisible distributions. The theorems of Cramér (Lévy-Cramér) and of Raikov on the factorization of the normal law and of the Poisson law, as well as various analytical extensions of Cramér's theorem, are proven. The theorems of Chapter 6 establish also the existence of a class I_0 of infinitely divisible distributions which have no indecomposable components. The main topic of the book concerns the investigation of the structure of the class I_0. The author studies first a special case (Chapter 7), namely the factorization of the convolution of Gauss and Poisson laws, and shows that this family is factor-closed. The theorems of Cramér and Raikov are both special cases of Linnik's result; the method of proof is however entirely different. This can be explained by the fact that Cramér's theorem deals with entire functions of finite order while Raikov's theorem concerns the characteristic function of a lattice distribution. Convolutions of normal and Poisson distributions have neither of these properties so that these advantages cannot be used, and more powerful analytical tools are needed for the study of the factorizations of these convolutions. At the same time the method employed can be expanded and can be used in the general investigation of the class I_0.

In order to discuss the results it is necessary to introduce certain terms. It is known that the characteristic function $\varphi(t)$ of an infinitely divisible distribution admits the representation

$$\begin{aligned}\log \varphi(t) = \beta it - \gamma t^2 + \textstyle\int_{-\infty}^{0} [e^{itu} - 1 - itu/(1 + u^2)]\, dM(u) \\ + \textstyle\int_0^\infty [e^{itu} - 1 - itu/(1 + u^2)]\, dN(u),\end{aligned} \tag{2}$$

where the functions $M(u)$ and $N(u)$ are non-decreasing and satisfy certain conditions. $M(u)$ is called the negative, $N(u)$ the positive, Poisson spectrum. One refers to u as the frequency and to $dM(u)$ and $dN(u)$ as the energy increments. The spectrum of $\varphi(t)$ (or of the corresponding distribution) is said to be bounded if there exists a constant A such that $\int_{-\infty}^{-A} dM(u) = \int_A^\infty dN(u) = 0$. A spectrum is called finite [or denumerable] if $dM(u) = dN(u) = 0$ outside a finite [or denumerable] set of frequencies. In this case one calls the increments of $M(u)$ [or of $N(u)$] at their points of increase the energy parameters of $M(u)$ [or $N(u)$].

In Chapter 8 a necessary condition is given which must be satisfied by a distribution which has a normal component [i.e., $\gamma > 0$ in (2)] in order that it should belong to I_0. The condition implies that the spectrum be either finite or denumerable and that the spectral frequencies have a special form. The proof of this condition is quite complicated and requires three lemmas which are of independent interest. These lemmas assert that certain functions are characteristic functions. Chapter 9 gives a sufficient condition for membership in I_0. The presence of a normal component is not required, otherwise the same conditions are imposed on the spectrum as in the previous chapter. Theorems about the form of the factors of infinitely divisible characteristic functions with bounded spectrum, as well as the factors of infinitely divisible characteristic functions whose spectrum is bounded inside but rational (i.e., the quotient of any two frequencies is a rational number) outside an interval are proven. Chapter 10 deals with a class of infinitely divisible distributions with unbounded spectrum; the distributions of this class have rapidly decreasing energy parameters and belong to I_0. Chapter 11 gives a brief discussion of stability theorems and of a general α-factorization theorem. A few classical results concerning infinitely divisible distributions without normal components are treated in Chapter 12. Cramér's theorem which describes a class of infinitely divisible distributions having indecomposable components is proven while results of P. Lévy and D. A. Raikov are stated without proof. Chapter 13 lists some interesting unsolved problems.

In this reviewer's opinion this is a very important book which contains a wealth of interesting material and which will for a long time provide stimulation to mathematicians and mathematical statisticians interested in analytical probability theory.

It is not the purpose of such a review to dwell upon the typography of a book.

However, the reviewer noticed with regret that the book is not produced in accordance with the high standards to which Gauthier-Villars has accustomed its readers. In addition to the misprints of the Russian edition, he found many new typographical errors. There are also some errors in the transliteration of the names of Western authors [for instance on p. 22, P. Debaiem for P. Debye; on p. 288, G. Cramér for H. Cramér] and in translation. For instance on pp. 160–162, СЧЕТНЫЙ is translated as "pair" (= even) while it should be "dénombrable" (= enumerable). In Lemma 2, (p. 163), НЕКОТОРЫЙ is translated as "quelconque" while on the preceding page, in Lemma 1, it is correctly rendered as "certaine" (= certain). The Landau symbol "O" is almost consistently printed as zero. This reviewer saw only a rough proof of the English translation but noted with satisfaction that the printing is far superior to the French edition. This proof contained comparatively few misprints and indicated a serious effort to correct obvious typographical errors of the original.

REFERENCE

[1] D. A. RAIKOV (1938). Factorization of the laws of Gauss and Poisson. *Izv. Akad. Nauk SSSR. Ser. Mat.* **2** 91–124.