BOOK REVIEWS

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ULF GRENANDER, *Probabilities on Algebraic Structures*. Wiley, New York and London, 1963, $12.00 and £4/5/10. 218 pp.

Review by G. KALLIANPUR

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In recent years, particularly within the last two decades, a great deal of work has been done in developing the theory of probability measures on topological groups and semi-groups, linear topological vector spaces (such as Banach spaces and the space of Schwartz distributions) and topological algebras. From a theoretical point of view this work can be regarded as an attempt to extend to these more general topological-algebraic structures the concepts and techniques that have proved so successful in studying Euclidean probability spaces, as well as to discover new ones. The impetus for such a theory has come both from its mathematical appeal and from a variety of (apparently unrelated) practical problems from communication engineering, physics and, not least, from statistics. The most significant contributions in this rapidly growing field are to be found widely scattered in mathematical journals and symposia and hitherto no unified account of these researches has been available to the interested student of probability and statistics. The present book is the first, necessarily tentative, effort in this direction, no doubt to be followed in the years to come by more ambitious and definitive studies.

The first chapter opens with a brief survey of the methods and results of the "classical" theory of real random variables, and proceeds to give a motivation for its extension by a discussion of several practical problems, followed by a short historical background. The author is anxious to convince the reader at the very start that, despite the highly abstract nature of the theory, its aim is not to generalize for the sake of generalization.

The subject proper is introduced in Chapter 2, where one begins by considering a space which is a topological semi-group (associative but not necessarily commutative) with probability measures defined on the Borel field generated by all open sets. There is now enough algebraic structure to define the convolution of probability measures and to launch the study of properties of products (or sums if the semi-group operation is thought of as addition) of independent random elements. The same chapter gets down to a more detailed study of compact semi-groups. A whole group of problems now unfolds itself: the limiting behavior of products of independent random elements, in particular, the law of large numbers and the central limit theorem, infinitely divisible distributions and homogeneous additive processes. This program is carried out in the next three chapters, which deal respectively with compact and locally compact commutative groups, Lie groups and general locally compact groups. Chapters 2 to 5 thus logically belong together and the reader begins to get a feeling for the type of problems that are important in this field as well as the difficulties—such as those created by non-commutativity—that are peculiar to the subject. The remaining two chapters on stochastic linear spaces (Chapter 6) and stochastic algebras (Chapter 7), although they fit into the general scheme of the book, somehow seem apart. Chapter 6 contains a succinct account of the theory of probabilities in Banach spaces as well as a brief section on random Schwartz distributions. The greater emphasis is naturally on Hilbert spaces. Here the author's aim seems to be to present a general summary rather than to go deeply into any single question. Sazonov's theorem giving necessary and sufficient conditions for a characteristic functional or a probability distribution in a separable Hilbert space is stated and proved in the text, but other recent work centering around these important ideas is relegated to the Notes. More serious, in my opinion, is the absence of any reference to the work of Skorohod, whose papers are not mentioned in an otherwise fairly extensive bibliography. This is all the more puzzling since the convergence of empirical distributions is considered in this context. It is true that, strictly speaking, it falls outside the scope of the book but its omission and the omission of limit theorems in other function spaces (e.g., convergence of probability measures to Wiener measure in $C[0, 1]$) deprives this chapter of much of its interest. Chapter 7 on stochastic algebras treats multiplicative as well as additive limit theorems and outlines very briefly the theory of random operators and equations and the elements of a stochastic spectral theory. The last section of each chapter except the first is reserved for worked examples. These are often simple special cases of the theory or interesting concrete problems having an important bearing on the theory. One wishes there were more problems of the latter category especially in the chapter on stochastic groups.

The proofs of theorems wherever they happen to be lengthy or complicated are given only in outline, the details being left to the reader. There are 24 pages of notes at the end of the book which contain clarification of points in the text, alternative proofs, and references to the relevant literature. There is no attempt (rightly, in my opinion) to present results in their fullest generality even where possible—the second countability axiom, for instance, is assumed wherever necessary. The style of presentation is informal (at times too much so). The overall effect, however, is to focus the attention of the reader on the basic ideas.

As the author himself points out there are some topics which do not receive sufficient emphasis in the book and others that are excluded. Among the latter are sample function properties of processes. Perhaps it should be noted here that there is another class of interesting and, from the point of view of applications, important problems that this book is not intended to cover. An example would be the theory of weakly stationary random fields on a group where the random variables of the field are real or complex valued and it is the indexing parameter set that is a topological group. Although this is not a case of probability "on" a general algebraic structure, the latter enters in an essential way (via stationarity) in the study of the process.

The reader of this review will have guessed by now that the book's mathematical prerequisites go considerably beyond a knowledge of standard measure theory and of advanced probability theory, say at the level of Loève's book. No secret is made of this fact in the Introduction, which expects that the student who wishes to work through the text should have familiarity with the elements of functional analysis (as in the well known book by Hille and Phillips) and topological algebra (as in Neumark's *Normed Rings*). I think that advanced graduate students intending to specialize in abstract probability theory will find this book a very useful text for seminars.

Almost all the topics treated in the book are areas in which active research is still being done, and the difficulties in presenting a logical development of the principal results in a fairly slim volume are many. I, for one, am glad that Professor Grenander has courageously undertaken the task.

YU. V. LINNIK, *Décomposition des lois de probabilités* (translated from the Russian by M. L. Gruel) (Monographies internationales de mathématiques modernes), Gauthier-Villars, Paris, 1962. Fr 55.—vi + 294 pp.

YU. V. LINNIK, *Decomposition of probability distributions* (edited by S. J. Taylor), Oliver and Boyd Ltd., London, 1964. £4/4/0.—239 pp.

Review by E. LUKACS

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One of the most important topics in classical probability theory is the addition of independent random variables, studied since the first quarter of the 18th century. The converse problem, the decomposition of a given random variable into independent summands, is of recent origin and was started by investigations of P. Lévy, A. Ya. Khintchine, H. Cramér, and D. A. Raikov during the late thirties of the present century. The last author raised a number of interesting questions [1] concerning the family of infinitely divisible laws which have no indecomposable components. No appreciable progress was made towards the solution of these problems until the studies of Yu. V. Linnik which were published in the Russian probability journal during the years 1957–1959. In the present monograph, the author gives a systematic and connected account of his investigations.

The theory of decomposition of random variables, often called the arithmetic of distribution functions, deals with problems on the borderline between probability theory and classical analysis. In order to make the presentation self-contained, the author presents in the first chapter a number of important re-