ON THE COMPLEX WISHART DISTRIBUTION

By M. S. Srivastava

University of Toronto

1. Introduction. Goodman [2] derived the complex Wishart distribution with the aid of characteristic functions and Fourier transforms. In the present paper we give a direct and simplified method of deriving this distribution. At the same time Lemma 13.3.1 of Anderson [1] is generalized to matrices with complex elements. This generalization leads to a straightforward extension of the results of Chapter 13 of Anderson [1] to complex matrices.

2. Preliminaries and definitions. For a complex number $z = x + iy$, $\bar{z}$ denotes the conjugate. A matrix M of elements m_{jk} is denoted by $\|m_{jk}\|$, the determinant of a square matrix by $|M|$, the transpose by M'.

For notational convenience, we shall not distinguish between a random variable and its observed values.

The pdf of a p-variate complex Gaussian distribution $\xi' = (z_1, z_2, \cdots, z_p)$ is

$$p(\xi) = \pi^{-p}|\Sigma_\xi|^{-1} \exp(-\bar{\xi}'\Sigma_\xi^{-1}\xi),$$

assuming each of the random variables x and y to have zero mean (See Goodman [2] for definitions).

Lemma 1. *If Y is a matrix of complex elements, of order $p \times m$, $p \leqq m$, and of rank p, then there exists a unique triangular matrix T with real and positive* (>0) *diagonal elements and a semi-unitary matrix L, $L\bar{L}' = I$, such that*

$$Y(p \times m) = T(p \times p)L(p \times m).$$

Proof. The proof is quite simple and is omitted.

3. Main results. The following theorem is a generalization of Lemma 13.3.1 of Anderson ([1], p. 319) to a matrix Y with complex elements.

Theorem 1. *If the density of $Y(p \times m)$ is $f(Y\bar{Y}')$, then the density of $B = Y\bar{Y}'$ is*

$$\{|B|^{m-p}f(B)\pi^{p[m-\frac{1}{2}(p-1)]}\}/[\textstyle\prod_{i=1}^{p}\Gamma(m-i+1)]. \tag{1}$$

Proof. From Lemma 1, we can write

$$Y = TL, \tag{2}$$

where T is a triangular matrix with positive (>0) diagonal elements, and L a semi-unitary matrix.

We shall now find the Jacobian of the transformation, $J(Y \rightarrow T, L)$. Differentiating both sides of Equation (2), we have $(dY) = (dT)L + T(dL)$. Premultiplying by T^{-1}, we get $T^{-1}(dY) = T^{-1}(dT)L + (dL)$. Putting $U = T^{-1}(dY)$, and $V = T^{-1}(dT)$, we have $U = VL + dL$.

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Hence the Jacobian of the transformation (2) is given by

$$\begin{aligned} J(Y \to T, L) &= J(dY \to dT, dL) \\ &= J(dY \to U)\cdot J(U \to V, dL)\cdot J(V, dL \to dT, dL) \\ &= J_1 \times J_2 \times J_3 \quad \text{(say)}. \end{aligned}$$

It is easy to check that

$$J_1 = |T|^{2m},$$
$$J_3 = \textstyle\prod_{i=1}^{p} (t_{ii}^{-2i+1}),$$

and J_2 is a function of L only, independent of T. Let us denote it by $J_2 = g(L)$. Hence the joint density of T and L is

$$g(L)\textstyle\prod_{i=1}^{p} (t_{ii}^{2(m-i)+1}) f(T\bar{T}').$$

We find, by integrating out L, that the density of T is

$$(3) \quad \textstyle\prod_{i=1}^{p} (t_{ii}^{2(m-i)+1}) f(T\bar{T}') \int_{L\bar{L}'=I} g(L)\, dL = C_1 \prod_{i=1}^{p} (t_{ii}^{2(m-i)+1}) f(T\bar{T}'),$$

where $C_1 = \int_{L\bar{L}'=I} g(L)\, dL$, a constant.

Making the transformation

$$\begin{aligned} B &= T\bar{T}' \\ &= Y\bar{Y}', \end{aligned}$$

we have the Jacobian of the transformation

$$2^{-p}\textstyle\prod_{i=1}^{p} (t_{ii}^{-2(p-i)-1}).$$

Hence the density of B is

$$(4) \quad \begin{aligned} C_2 \textstyle\prod_{i=1}^{p} (t_{ii}^{2(m-p)}) f(B) &= C_2 |T\bar{T}'|^{m-p} f(B) \\ &= C_2 |B|^{m-p} f(B), \end{aligned}$$

where $C_2 = 2^{-p}C_1$, a constant. The constant is evaluated in the next section.

Alternative proof. This is done by obtaining the distribution in two ways and then comparing them.

First. Make transformations $Y = AX$, $V = X\bar{X}'$, and then $B = AV\bar{A}'$. The distribution of B is

$$(5) \quad |A|^{2(m-p)} f(B) h(A^{-1}B\bar{A}'^{-1}),$$

where $h(V)$ is the Jacobian of the transformation from X to V.

Second. Make a transformation $Y\bar{Y}' = B$. Then the distribution of B is

$$(6) \quad f(B)h(B).$$

Hence, comparing (5) and (6), we have $h(B) = |B|^{m-p}h(I)$, and hence the density of B is Const. $|B|^{m-p}f(B)$.

Theorem 2. *If $A \sim W_c(\Sigma, p, n)$, and $B \sim W_c(\Sigma, p, m)$ are independently* *distributed, then* $A + B \equiv W\bar{W}'$ *and* $Z = W^{-1}A\bar{W}'^{-1}$ *are statistically independent. Furthermore, the distribution of* Z *is invariant under the transformation* $Z \to \Gamma Z\bar{\Gamma}'$, *where* Γ *is unitary.*

The proof of this theorem is straightforward and will be omitted.

The analogue of Theorem 2 of Olkin and Rubin [3] for the characterization of the complex Wishart distribution is under investigation.

4. Evaluation of the constant in (4). From (3), the density of T is given by

$$p(T) = C_1 \prod_{i=1}^{p} t_{ii}^{2(m-i)+1} f(T\bar{T}').$$

Let f be a standard multivariate complex Gaussian distribution. In this case $p(T)$ becomes

$$C_1(\pi^{mp})^{-1} \prod_{i=1}^{p} (t_{ii}^{2(m-i)+1}) \exp - \operatorname{tr} T\bar{T}'.$$

To evaluate the constant, C_1, we know that

$$\begin{aligned} 1 &= C_1(\pi^{mp})^{-1} \int \prod_{i=1}^{p} (t_{ii}^{2(m-i)+1}) (\exp - \operatorname{tr} T\bar{T}')\, dT \\ &= C_1 2^{-p} \prod_{i=1}^{p} \Gamma(m - i + 1) \pi^{p[\frac{1}{2}(p-1)-m]}. \end{aligned}$$

Therefore the constant C_1 in (3) is equal to $2^p / \prod_{i=1}^{p} \Gamma(m - i + 1)\pi^{p[\frac{1}{2}(p-1)-m]}$. Hence the value of the constant C_2 in (4) is equal to

$$1 / \prod_{i=1}^{p} \Gamma(m - i + 1)\, \pi^{p[\frac{1}{2}(p-1)-m]}.$$

The complex Wishart distribution is thus obtained by substituting for f in (4), the multivariate complex Gaussian distribution.

REFERENCES

[1] Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis*. Wiley, New York.

[2] Goodman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution. (An introduction) *Ann. Math. Statist.* **34** 152–177.

[3] Olkin, I. and Rubin, H. (1962). A characterization of the Wishart distribution. *Ann. Math. Statist.* **33** 1272–1280.