ON LIMIT THEOREMS FOR GAUSSIAN PROCESSES[1]

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1. Introduction. This paper has two purposes. It consists of a new attack, using different methods, on some problems concerning large excursions of Gaussian processes which were studied by M. Kac and D. Slepian in [1], and it may be considered a technical contribution to this topic since some new and perhaps simpler proofs are given under less restrictive assumptions than theirs. These problems (among others) have also been studied by Volkonski and Rozanov in [4], but again under more stringent assumptions than we shall require. On the other hand, this paper is intended to provide a further illustration of the point made in [2] that choice of the proper topology may be of great importance in connection with convergence of stochastic processes. It seems likely that the method used below will have other applications in the study of Gaussian processes, and it is this hope which is responsible for the rather general title I have chosen.

2. The limiting process. Let $\{x(t)\}$ denote a real, continuous, stationary, Gaussian stochastic process with mean 0 and covariance function satisfying

$$\rho(t) = E(x(s)x(t+s)) = 1 - \tfrac{1}{2}\alpha t^2 + o(t^2) \tag{1}$$

for small t. Kac and Slepian studied the behavior of $x(t)$, conditioned so that $x(0) = a$, as a tends to $+\infty$; the particular question of greatest interest is the distribution of the time until the next return to the level a in the cases when $x'(0) > 0$. They also discussed several different interpretations of the conditional probabilities, which led to somewhat different results. In this paper, however, we shall work at first with the ordinary notion of conditional probabilities and densities (equivalent to those called "vertical window" in [1]); this considerably simplifies the problem since the conditioned processes are then still Gaussian. Later on it will be shown how the results can all be carried over to the other types of conditioning considered by Kac and Slepian.

Let us now define

$$\Delta(t, \theta) = (x(\theta t) - a)/\theta, \tag{2}$$

where

$$\theta = (2\pi/\alpha)^{\frac{1}{2}} a^{-1} \quad \text{for} \quad a > 0; \tag{3}$$

obviously $\Delta(0, \theta) = 0$ and $\theta \to 0$ are respectively equivalent to $x(0) = a$ and

Received 14 August 1964.

[1] This research was supported by the National Science Foundation.

$a \to \infty$. Define also a process $\{z(t)\}$ by putting

$$z(t) = -(\tfrac{1}{2}\alpha\pi)^{\frac{1}{2}}t^2 + \alpha^{\frac{1}{2}}t\xi, \tag{4}$$

where ξ is a random variable with a standard normal distribution. We will first show that *the finite-dimensional distributions of* $\{\Delta(t, \theta)\}$, *given that* $\Delta(0, \theta) = 0$, *converge as* $\theta \to 0$ to those of $\{z(t)\}$.

The proof is very easy; since $\{\Delta(t, \theta)\}$, conditioned by $\Delta(0, \theta) = 0$, is still a Gaussian process it is only necessary to show that its mean and covariance functions converge to those of $\{z(t)\}$. Either "by hand" or using the results in the appendix of [1], we readily obtain

$$E(\Delta(t, \theta) \mid \Delta(0, \theta) = 0) = (a/\theta)[\rho(\theta t) - 1], \tag{5}$$

$$\text{Cov}\,(\Delta(t, \theta), \Delta(s, \theta) \mid \Delta(0, \theta) = 0) = \theta^{-2}[\rho(\theta(t - s)) - \rho(\theta t)\rho(\theta s)]. \tag{6}$$

Using (1) and (3) it is immediate that as $\theta \to 0$ these quantities have the limits $-(\frac{1}{2}\alpha\pi)^{\frac{1}{2}}t^2$ and αst respectively, which are the mean and covariance functions of $\{z(t)\}$.

For any continuous function $y(t)$ with $y(0) = 0$, define the functional

$$T(y) = \inf\{t > 0 : y(t) = 0\}. \tag{7}$$

Thus $\Pr(T(\Delta(t, \theta)) \leqq \tau \mid \Delta(0, \theta) = 0)$ is the conditional probability that $x(t)$ has a positive a-crossing before time $\theta\tau$, given that $x(0) = a$. We seek a limiting distribution as $\theta \to 0$; in view of the result above it is natural to try to prove that the limit is $\Pr(T(z(t)) \leqq \tau)$, which is easily written down explicitly using (4).

This is the sort of result that is often proved by an "invariance principle" type of argument, although it was not so approached in [1]. The difficulty is that the functional T is *not* continuous under uniform convergence of y_n to y for any y such that $T(y) > 0$, since the approximating functions may have rapid small oscillations near 0, forcing $T(y_n) \to 0$, and still be uniformly convergent to y. The "invariance" method can be applied, however, by utilizing a stronger topology on function space than the uniform one. We turn next, accordingly, to a brief study of convergence of Gaussian processes on a space of differentiable functions.

3. Convergence in C^1. Let $\{y^{(n)}(t)\}$ be a sequence of separable, Gaussian random processes with mean functions $\mu_n(t)$ and covariances $\rho_n(s, t)$; the parameter interval is $[0, A]$, and $y_n(0) = 0$. Assume that

$$\lim_{n\to\infty} \mu_n(t) = \mu(t) \quad \text{and} \quad \lim_{n\to\infty} \rho_n(s, t) = \rho(s, t). \tag{8}$$

Let C^1 denote the Banach space of real functions on $[0, A]$ with continuous derivatives; the norm is the maximum of the function plus that of the derivative. We will use the following

THEOREM. *Suppose that, for each* n, $\mu_n'(t)$ *and* $q_n(s, t) = [\partial^2\rho_n(s, t)]/(\partial s\, \partial t)$ *exist for* $0 \leqq s, t \leqq A$. *Suppose further that the functions* $\mu_n'(t)$ *are continuous and converge uniformly and that the inequality*

$$(9) \qquad |q_n(t, t) - 2q_n(t, s) + q_n(s, s)| \leqq B|t - s|^{\delta}$$

holds for all $s, t \in [0, A]$ *with the choice of* B *and* $\delta > 0$ *independent of* n. *Then there exists a Gaussian process* $\{y(t)\}$ *with mean* μ, *covariance* ρ, *and* a.s. *continuously differentiable paths, such that the distribution function of* $f(y^{(n)}(t))$ *converges weakly to that of* $f(y(t))$ *for every real-valued functional on* C^1 *which is Borel-measurable and continuous at almost all paths of* $\{y(t)\}$.

PROOF. This result is an easy application of a theorem of Prokhorov [3, Thm. 2.1] (see also [2]), which states that if the condition

$$(10) \qquad E(|\xi_n(t) - \xi_n(s)|^{\alpha}) \leqq B|t - s|^{1+\beta}, \qquad \alpha, \beta > 0,$$

holds uniformly in n for a sequence of separable stochastic processes $\{\xi_n(t)\}$, and if their finite-dimensional distributions have limits as $n \to \infty$, then the measures induced by the processes on the Banach space C of continuous functions on $[0, A]$ converge weakly to the measure of a process with the limiting finite-dimensional laws. In case the processes are Gaussian with zero means, it is easily seen that the validity of a condition of the form (9) for their covariances implies that a condition (10) holds for some positive β.

We shall apply this result to the processes $\{y_n'(t) - \mu_n'(t)\}$. In fact the means are 0 and the covariances of these processes are $q_n(s, t)$ which satisfy (9) by assumption; we also have

$$\lim_{n\to\infty} q_n(s, t) = [\partial^2 \rho(s, t)]/(\partial s\, \partial t)$$

as a consequence of (8) and (9). It follows that the measures of $\{y_n'(t) - \mu_n'(t)\}$ converge weakly on C. Since our assumptions imply that $\lim_{n\to\infty} \mu_n'(t) = \mu'(t)$ uniformly on $[0, A]$, the same conclusion holds for the processes $\{y_n'(t)\}$ themselves. Let us call the limit process $\{y'(t)\}$; its integral, with $y(0) = 0$, has mean μ and covariance ρ.

An equivalent formulation of the above fact is that if f is any real functional continuous everywhere on C, then the law of $f(y_n'(t))$ converges weakly to that of $f(y'(t))$. But suppose that g is a continuous functional on C^1; defining

$$f(\xi(t)) = g(\textstyle\int_0^t \xi(\tau)\, d\tau)$$

for every continuous function $\xi(t)$, we have

$$g(y_n(t)) = f(y_n'(t)).$$

Also, it is clear that C^1 continuity of g implies the continuity of f with respect to the uniform metric. Hence the law of the random variable $g(y_n(t))$ converges to that of $g(y(t))$; in other words, the measures induced on C^1 by $\{y_n(t)\}$ converge weakly to that induced by $\{y(t)\}$. The conclusion of our theorem is a standard consequence of such convergence.

REMARK. The conclusion of the theorem can be somewhat strengthened by using the sharpening of Prokhorov's theorem which is presented in [2]; the advantages of doing so do not seem at present to be very significant. Alternatively, Assumption (9) can probably be weakened.

4. C^1 convergence of $\Delta(t, \theta)$. We now apply the theorem of Section 3 to the convergence of $\{\Delta(t, \theta)\}$, conditioned by $x(0) = a$, to the limiting process $\{z(t)\}$. The result is as follows:

THEOREM. *Suppose that Condition* (1) *is strengthened to*

$$\rho(t) = 1 - \tfrac{1}{2}\alpha t^2 + O(t^{2+\epsilon}), \qquad \epsilon > 0. \tag{11}$$

Then for any Borel-measurable functional f on C^1 which is continuous at almost all of the parabolas $\{z(t)\}$ we have

$$\lim_{\theta\to 0} \Pr(f(\Delta(t, \theta)) \leqq u \mid \Delta(0, \theta) = 0) = \Pr(f(z(t)) \leqq u) \tag{12}$$

at values of u such that the right hand side is continuous.

We will carry out the proof in several easy stages.

LEMMA 1. *Assuming only Condition* (1) *we have*

$$\lim_{\theta\to 0} (d/dt)E(\Delta(t, \theta) \mid \Delta(0, \theta) = 0) = -2(\tfrac{1}{2}\alpha\pi)^{\frac{1}{2}}t, \tag{13}$$

and the convergence is uniform for $0 \leqq t \leqq A < \infty$.

PROOF. The covariance function $\rho(t)$ has the representation

$$\rho(t) = \int_0^\infty \cos tx \, dF(x), \tag{14}$$

and it follows from (1) that $\alpha = \int_0^\infty x^2 \, dF(x)$ is finite. Accordingly, using (5) and (3) we can write

$$E(\Delta(t, \theta) \mid \Delta(0, \theta) = 0) + (\tfrac{1}{2}\alpha\pi)^{\frac{1}{2}}t^2 = \left(\frac{2\pi}{\alpha}\right)^{\frac{1}{2}} \int_0^\infty \left\{\frac{\cos \theta tx - 1}{\theta^2} + \frac{t^2x^2}{2}\right\} dF(x).$$

As a result we have

$$\frac{d}{dt} E(\Delta(t, \theta) \mid \Delta(0, \theta) = 0) + 2(\tfrac{1}{2}\alpha\pi)^{\frac{1}{2}}t = \left(\frac{2\pi}{\alpha}\right)^{\frac{1}{2}} \int_0^\infty tx^2 \left\{1 - \frac{\sin \theta tx}{\theta tx}\right\} dF(x). \tag{15}$$

Let the integral on the right side be written as $\int_0^M + \int_M^\infty$. Since $|(\sin \omega)/\omega| \leqq 1$ for all ω, the second term is at most $2A\int_M^\infty x^2 \, dF(x)$, which is small when M is large uniformly in θ. In $\int_0^M$, on the other hand the integrand tends to 0 with θ uniformly in x and t over their stated ranges. Hence the right side of (15) goes to 0 with θ uniformly in $t(\leqq A)$, and Lemma 1 is proved.

LEMMA 2. *Condition* (11) *implies that $\int_0^\infty x^{2+\epsilon'} \, dF(x) < \infty$ for all $\epsilon' < \epsilon$.*

PROOF. Let us put

$$h(t) = \rho(t) - 1 + \tfrac{1}{2}\alpha t^2 = \int_0^\infty \{\cos tx - 1 + \tfrac{1}{2}t^2x^2\} \, dF(x). \tag{16}$$

Let $f(\omega) = \cos \omega - 1 + \frac{1}{2}\omega^2$; it is easy to see that $f(\omega) > 0$ for all $\omega > 0$ and f is continuous. Consequently for any $t > 0$ we have

$$h(t) = \int_0^\infty f(tx) \, dF(x) \geqq \int_{1/t}^{2/t} f(tx) \, dF(x) \geqq \min_{1 \leqq u \leqq 2} f(u)\{F(2/t) - F(1/t)\}.$$

Putting $t = 2^{-n}$, the Assumption (11) that $h(t) = O(t^{2+\epsilon})$ yields

$$F(2^{n+1}) - F(2^n) \leqq C2^{-(2+\epsilon)n}.$$

From this estimate it follows at once that all moments of F of order less than $2 + \epsilon$ must be finite

LEMMA 3. *Condition* (11) *implies that the mixed second partial derivative of the conditional covariance function of* $\{\Delta(t, \theta)\}$, *given* $\Delta(0, \theta) = 0$, *satisfies a condition of the form* (9) *uniformly in* θ.

PROOF. The covariance function in question was given in (6); the mixed partial derivative is accordingly

$$q_\theta(t, s) = -\rho''(\theta(t - s)) - \rho'(\theta t)\rho'(\theta s). \tag{17}$$

It is of course enough to obtain (9) separately for each of the two terms in (17). For the second term, however, this is trivial (with $\delta = 2$) in view of the fact that ρ'' is bounded. Condition (9) for the first term in (17) becomes

$$|-\rho''(0) + \rho''(\theta(t - s))| \leqq B|t - s|^\delta, \qquad \delta > 0, \tag{18}$$

to hold for all small θ; clearly it is enough to obtain (18) with $\theta = 1$, for all $|\omega| = |t - s| \leqq A$. Using the spectral representation (14) of $\rho(\omega)$, plus the existence of the second moment of F, we can write

$$-\rho''(0) + \rho''(\omega) = \int_0^\infty (1 - \cos \omega x)x^2 \, dF(x). \tag{19}$$

Choose $\eta \in (0, \min(\epsilon, 1))$, so that by the previous lemma $\int_0^\infty x^{2+\eta} \, dF(x)$ is finite. Since $1 - \cos u \leqq Cu^2$ for all u, we can write

$$0 \leqq \int_0^{\omega^{\eta-1}} (1 - \cos \omega x)x^2 \, dF(x) \leqq C\omega^2 \int_0^{\omega^{\eta-1}} x^4 \, dF(x)$$
$$\leqq C\omega^2(\omega^{\eta-1})^2 \int_0^{\omega^{\eta-1}} x^2 \, dF(x) \leqq C'\omega^{2\eta}.$$

For the remainder of the range of integration we have

$$0 \leqq \int_{\omega^{\eta-1}}^\infty (1 - \cos \omega x)x^2 \, dF(x) \leqq \int_{\omega^{\eta-1}}^\infty x^2 \, dF(x)$$
$$\leqq \int_{\omega^{\eta-1}}^\infty x^2(x/\omega^{\eta-1})^\eta \, dF(x) \leqq \omega^{\eta(1-\eta)} \int_0^\infty x^{2+\eta} \, dF(x).$$

Combining these two estimates with (19), we obtain (18) with $\delta = \eta(1 - \eta) > 0$.

PROOF OF THE THEOREM. The results of Section 2 about convergence of mean and covariance functions, plus Lemmas 1 and 3, verify all the hypotheses of the theorem of Section 3; the conclusion specializes to the result stated above.

COROLLARY. *Under* (11), *for any* $\tau > 0$ *we have*

$$\lim_{\theta \to 0} \Pr(\Delta(t, 0) > 0 \quad \text{for} \quad 0 < t \leqq \tau \mid \Delta(0, \theta) = 0) = 1 - \Phi((\tfrac{1}{2}\pi)^{\frac{1}{2}}\tau), \tag{20}$$

where Φ *is the standard normal distribution.*

PROOF. The functional T defined in (7), modified to have value A if $y(t)$ has no zero in $(0, A)$, is continuous on C^1 at all paths of $\{z(t)\}$ except the one with $z'(0) = 0$. Accordingly the conditional distribution of $T(\Delta(t, \theta))$ given $\Delta(0, \theta) = 0$ converges to the law of $T(z(t))$, and (20) is the result. (Use $A > \tau$.)

5. Other types of conditioning. Even in the "vertical window" case Kac and Slepian's results are slightly different from ours, since they condition by putting

$\Delta(0, \theta) = 0$ *and* $\Delta'(0, \theta) = x'(0) \geqq 0$, thus ensuring that an "upcrossing" takes place initially. However, our theorem easily yields the more general formula

$$\lim_{\theta\to 0} \Pr(f(\Delta(t, \theta)) \leqq u \mid \Delta(0, \theta) = 0, c \leqq \Delta'(0, \theta) \leqq d) = \Pr(f(z(t)) \leqq u \mid c \leqq z'(0) \leqq d) \tag{21}$$

(for u a continuity point) for functionals f which are continuous on C^1. To prove (21), we rewrite the conditional probability as

$$\frac{\Pr(f(\Delta(t, \theta)) \leqq u, c \leqq \Delta'(0, \theta) \leqq d \mid \Delta(0, \theta) = 0)}{\Pr(c \leqq \Delta'(0, \theta) \leqq d \mid \Delta(0, \theta) = 0)}. \tag{22}$$

Now the event $\{f(\Delta(t, \theta)) \leqq u, c \leqq \Delta'(0, \theta) \leqq d\}$ defines a Borel subset of C^1 whose boundary has measure 0 with respect to the process $\{z(t)\}$. Therefore it follows from the weak convergence which was established in the previous section that the conditional probability of this event given $\Delta(0, \theta) = 0$ tends to $\Pr(f(z(t)) \leqq u, c \leqq z'(0) \leqq d)$. The probabilities in the denominator of (22) tend to $\Pr(c \leqq z'(0) \leqq d)$ for the same reason. (These probabilities are actually equal to $\Pr(c \leqq x'(0) \leqq d)$ and so independent of θ.) Combining these facts gives (21).

Next we will obtain an easy generalization of (21) which allows for, among others, "horizontal window" conditioning.

THEOREM. *For the same functionals f as before, we have*

$$\lim_{\theta\to 0} \int_{-\infty}^{\infty} \Pr(f(\Delta(t, \theta)) \leqq u \mid \Delta(0, \theta) = 0, \Delta'(0, \theta) = y)p(y)\, dy = \int_{-\infty}^{\infty} \Pr(f(z(t)) \leqq u \mid z'(0) = y)p(y)\, dy, \tag{23}$$

where $p(y)$ is any continuous probability density.

We remark that the obvious line of attack—showing convergence of the integrands—can not be used without imposing a stricter condition on $\rho(t)$ than that in (11). We will instead obtain (23) from (21) using the following

LEMMA. *Suppose for $0 \leqq \theta \leqq 1$ that $g_\theta(y)$ is measurable in y and satisfies $0 \leqq g_\theta(y) \leqq 1$. Suppose also that for all $-\infty \leqq c < d \leqq +\infty$,*

$$\lim_{\theta\to 0} \int_c^d \exp(-y^2/2\alpha) g_\theta(y)\, dy = \int_c^d \exp(-y^2/2\alpha) g_0(y)\, dy. \tag{24}$$

Then

$$\lim_{\theta\to 0} \int_c^d g_\theta(y) p(y)\, dy = \int_c^d g_0(y) p(y)\, dy \tag{25}$$

for any $p(y)$ which is continuous and absolutely integrable on $(-\infty, \infty)$.

PROOF. Consider measures μ_θ defined by the densities $\exp(-y^2/2\alpha)g_\theta(y)$; (24) implies that μ_θ converges weakly to μ_0 as $\theta \to 0$. From this, and the uniform boundedness of $g_\theta(y)$, it follows easily that

$$\lim_{\theta\to 0} \int_c^d h(y)\, d\mu_\theta(y) = \int_c^d h(y)\, d\mu_0(y)$$

for any continuous function h and finite interval (c, d). Choosing $h(y) = p(y) \exp(y^2/2\alpha)$ we have (25) for finite intervals; using the integrability of p and again the uniform bound on $g_\theta(y)$ it is easy to see that this suffices.

To prove the theorem, for $\theta > 0$ we let

$$\Pr(f(\Delta(t, \theta)) \leqq u \mid \Delta(0, \theta) = 0, \Delta'(0, \theta) = y) = g_\theta(y). \tag{26}$$

Now $\Delta'(0, \theta) = x'(0)$ is a normally distributed random variable with mean 0 and variance α, and so we have

$$\begin{aligned}\Pr(f(\Delta(t, \theta)) &\leqq u \mid \Delta(0, \theta) = 0, c \leqq \Delta'(0, \theta) \leqq d) \\ &= (2\pi\alpha)^{-\frac{1}{2}} \int_c^d \exp(-y^2/2\alpha) \Pr(f(\Delta(t, y) \leqq u \mid \Delta(0, \theta) = 0, \Delta'(0, \theta) = y)\, dy \\ &= (2\pi\alpha)^{-\frac{1}{2}} \int_c^d \exp(-y^2/2\alpha) g_\theta(y)\, dy.\end{aligned}$$

By (21), as θ approaches 0 this tends to

$$\Pr(f(z(t)) \leqq u \mid c \leqq z'(0) \leqq d) = (2\pi\alpha)^{-\frac{1}{2}} \int_c^d \Pr(f(z(t)) \leqq u \mid z'(0) = y) \exp(-y^2/2\alpha)\, dy.$$

Accordingly if we set $g_0(y) = \Pr(f(z(t)) \leqq u \mid z'(0) = y)$ relation (24) is satisfied. The conclusion of the lemma then yields (23).

As an illustration we will give explicitly the results for "horizontal-window" conditioning. As explained in [1],

$$\begin{aligned}\Pr(f(\Delta(t, \theta)) &\leqq u \mid \Delta'(0, \theta) \geqq 0, \Delta(0, \theta) = 0)_{\text{h.w.}} \\ &= \alpha^{-1} \int_0^\infty y \exp(-y^2/2\alpha) \Pr(f(\Delta(t, \theta)) \leqq u \mid \Delta'(0, \theta) = y, \Delta(0, \theta) = 0)\, dy.\end{aligned}$$

By (23), this quantity has for $\theta \to 0$ the limit

$$\alpha^{-1} \int_0^\infty y \exp(-y^2/2\alpha) \Pr(f(z(t)) \leqq u \mid z'(0) = y)\, dy.$$

Specializing to the functional T (see the corollary in Section 4 and its proof), it is straight-forward to evaluate the limit explicitly and obtain

$$\begin{aligned}&\lim_{\theta\to 0} \Pr(\Delta(t, \theta) > 0 \quad \text{for} \\ &\qquad 0 < t < \tau \mid \Delta'(0, \theta) \geqq 0, \Delta(0, \theta) = 0)_{\text{h.w.}} = \exp(-\tfrac{1}{4}\pi\tau^2)\end{aligned} \tag{27}$$

which agrees with equation (4.3) of [1]. Kac and Slepian's method also gives convergence of the densities, a result which does not follow from our theorems without further work. On the other hand, the condition (11) used in our derivation is considerably more general than (4.1) of [1], and in addition the limiting distribution has here been found for many other functionals in addition to T.

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