# JOINT DISTRIBUTIONS WITH PRESCRIBED MOMENTS[1]

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**1. Introduction.** Suppose two random variables $X$ and $Y$ have a joint distribution, whereas only the marginal distributions are known, and also certain other information. (For example, let the two random variables represent height from reference level at two randomly chosen points a given distance apart on the Moon's surface. The distribution of heights at a single point is fairly well-known, and radar data also determine the average value of the absolute value of the difference in heights, that is, $E(|X - Y|)$, [5].) It is then desired to make inferences about the joint distribution of $X$ and $Y$. This article adopts the principle of maximum entropy [2] to select one plausible joint distribution from the various possible ones consistent with the given marginal distributions and a given set of cross moments. It is shown that a joint distribution of maximum entropy exists and is unique, under very general conditions. The unique joint distribution is shown to have a special form, by means of which it is shown that the unique joint distribution can be obtained as the solution of a certain set of non-linear integral equations. The techniques involve functional analysis and the calculus of variations, as well as standard properties of the entropy functional.

**2. Problem statement.** We shall solve problems of the following kind. We are given two random variables $X$ and $Y$ and information about their joint density function $f = f(x, y)$. In particular, we are given the marginal density functions of $X$ and $Y$, say $p(x)$ and $q(y)$. Thus,

$$\int_{-\infty}^{\infty} f(x, y)\, dy = p(x), \qquad \int_{-\infty}^{\infty} f(x, y)\, dx = q(y),$$

where both equations hold almost everywhere (a.e.). In addition, we are given a finite number (possibly zero) of other linear constraints which the density function $f$ must satisfy. These constraints are to be expressed in the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) r_j(x, y)\, dx\, dy = \rho_j, \qquad j = 1, \cdots, k,$$

where the $r_j$ are given functions and the $\rho_j$ are given constants. The problem is to find that joint density function $f$ which satisfies all these constraint equations and which in addition has maximum entropy

$$H(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f \log f^{-1}\, dx\, dy.$$

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We shall show in Theorem 1 that under quite general conditions there is a unique (up to sets of measure zero) such function $f$, and in Theorem 2 we prove that this function $f$ has a very special form which is enough to characterize it uniquely.

**3. Definitions.** We begin with the (ad hoc) definition of *an admissible set of functions.*

DEFINITION 1. Let $p(x)$ and $q(y)$ be two fixed density functions defined in the interval $(-\infty, \infty)$. Let $r_1(x, y), \cdots, r_k(x, y)$ be fixed, real valued functions defined in the $(x, y)$-plane. We shall always suppose that no non-zero linear combination of the $r_j(x, y)$ is equal almost everywhere to a function of $x$ plus a function $y$, since the expected values of such sums are determined solely by the marginal distributions and have no influence on the maximization problem. Let $\rho_1, \cdots, \rho_k$ be given constants. Let $F = F(p, q; r_1, \cdots, r_k; \rho_1, \cdots, \rho_k)$ be the set of density functions $f = f(x, y)$ which satisfy the following conditions:

$$(\alpha) \qquad \int_{-\infty}^{\infty} f(x, y)\, dx = q(y);$$

$$(\beta) \qquad \int_{-\infty}^{\infty} f(x, y)\, dy = p(x);$$

$(fr_j) \,\varepsilon\, L_1$ for $j = 1, \cdots, k$ with

$$(\gamma) \qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y) r_j(x, y)\, dx\, dy = \rho_j,$$

where Equations $(\alpha)$ and $(\beta)$ hold everywhere and $L_1$ is the set of Lebesgue integrable functions (in the $(x, y)$-plane in this case, but we shall also use $L_1$ to denote the Lebesgue integrable functions on the line as well). We shall call such an $F$ an admissible set of functions if the following statements are true in addition:

(a) $F$ is nonempty, and there is at least one function $f \,\varepsilon\, F$ such that $f \log f \,\varepsilon\, L_1$.

(b) $p \log p$, $q \log q \,\varepsilon\, L_1$.

(c) If $f \,\varepsilon\, F$, then $\int\int_{f<1} f \log^2 f \leqq A$, where $A$ is a constant (depending on $F$),

(d) $F$ is closed in $L_1$.

It is not apparent from this definition how one could decide whether a given set $F = F(p, q; r_1, \cdots, r_k; \rho_1, \cdots, \rho_k)$ is admissible or not. For example, it might be very difficult to decide whether $F$ is nonempty or not. However, this class of problem has been dealt with extensively in the literature [1], and we shall not discuss it further in this paper. The other Conditions (b), (c) and (d) are more easily dealt with. In fact, (b) is trivially checkable, since $p$, $q$ are given, and (c) can be made to depend on some weak condition on $p$ and $q$. Thus, for example, we have the following:

LEMMA 1. *Let $p(x)$ and $q(y)$ be density functions such that $xp(x)$, $yq(y) \,\varepsilon\, L_1$ (that is, the means of $X$, $Y$ exist). Let $f = f(x, y)$ be a joint density function with* *marginal densities* $p(x)$ *and* $q(y)$. *Then*

$$\iint_{f<1} f \log^2 f < A, \quad \textit{where } A \textit{ depends only on } p \textit{ and } q.$$

PROOF.

$$\begin{aligned}
\iint_{f<1} f \log^2 f &= \iint_{\left\{\substack{f<1 \\ \log^2 f > |x|+|y|+1}\right\}} f \log^2 f + \iint_{\left\{\substack{f<1 \\ \log^2 f \leqq |x|+|y|+1}\right\}} f \log^2 f \\
&\leqq \iint_{-\infty}^{\infty} \{\exp - (|x| + |y| + 1)^{\frac{1}{2}}\}(|x| + |y| + 1) \\
&\quad + \iint_{-\infty}^{\infty} (|x| + |y| + 1) f \\
&\leqq A_1 + \int_{-\infty}^{\infty} |x|\, p(x)\, dx + \int_{-\infty}^{\infty} |y|\, q(y)\, dy + 1 \\
&\leqq A,
\end{aligned}$$

where $A$ depends only on $p$ and $q$. This proves Lemma 1. It is thus evident from this lemma that Condition (c) of the definition of an admissible set of functions can be made to depend on some type of restriction on the marginal density functions $p$ and $q$.

Similarly, (d) can also be made to depend upon special properties of the function $r_1, \cdots, r_k$, and $p$ and $q$. In fact, let $f_1, \cdots, f_n, \cdots$, be a convergent sequence of functions in $F$ and let $f$ be the limit function in $L_1$. In order to verify that (d) is true we must show that $f$ satisfies Equations ($\alpha$), ($\beta$), and ($\gamma$). But Equations ($\alpha$), ($\beta$) follow immediately from the $L_1$ convergence of $\{f_n\}$ to $f$ (by taking Fourier transforms for example). Thus we need only verify that

$$\iint f(x, y) r_j(x, y) = \rho_j, \qquad j = 1, \cdots, k. \tag{1}$$

If the function $r_j$ in Equation (1) is in $L_\infty$ (the set of essentially bounded Lebesgue measurable functions) then Equation (1) is an immediate consequence of the $L_1$ convergence of the sequence $\{f_n\}$ to $f$. If $r_j(x, y)$ is not essentially bounded but satisfies some inequality of the form

$$|r_j(x, y)| \leqq \sum_{i=1}^{n} R_i(x) S_i(y),$$

then we can guarantee the validity of Equation (1) by stipulating that $p(x)$ and $q(y)$ are such that for $i = 1, \cdots, n$ either $R_i^2 p$ and $S_i^4 q \in L_1$, or $R_i^4 p$ and $S_i^2 q \in L_1$. To prove this statement it is only necessary to apply the Schwarz inequality several times to the difference

$$\left(\iint fr_j\right) - \rho_j = \iint (f - f_n)r_j .$$

The point of the discussion is that, although it may be exceedingly difficult to decide in complete generality whether the set $F = F(p, q; r_1, \cdots, r_k; \rho_1, \cdots, \rho_k)$ is admissible, in particular cases the admissibility of $F$ can be affirmed by some weak constraints on $p, q; r_1, \cdots, r_k; \rho_1, \cdots, \rho_k$, except for Condition (a) perhaps. But once an $f$ is found by the methods of Section 6, then even this problem will be taken care of.

We shall need the following

DEFINITION 2. Let $f = f(x, y)$ be a density function on the $(x, y)$-plane. We define the *entropy* of $f$ (if it exists), $H(f)$, by the equation

$$H(f) = \iint f \log f^{-1}.$$

We also define two operators $H_+$ and $H_-$ as follows:

$$H_+(f) = \iint_{f<1} f \log f^{-1},$$

$$H_-(f) = \iint_{f>1} f \log f.$$

It is evident that $H_+(f)$ and $H_-(f)$ are positive and that

$$H(f) = H_+(f) - H_-(f),$$

if $H(f)$, or $H_+(f)$ and $H_-(f)$, exist (in $L_1$).

**4. Existence and uniqueness.** We now state the first theorem of this paper.

THEOREM 1. *Let $F = F(p, q; r_1, \cdots, r_k; \rho_1, \cdots, \rho_k)$ be an admissible set. Then there exists a function $f \varepsilon F$ such that $H(f)$ is a maximum. Furthermore, $f$ is unique (to within sets of measure zero).*

PROOF. As the proof of this theorem depends on a number of lemmas, we shall begin with an outline of the successive steps in the proof. We first show that the entropies $H(f)$ of functions $f \varepsilon F$ are bounded from above. We call the least upper bound of these entropies $H_{\max}$. Then we show that if $f$ and $g$ are two functions in $F$ with entropies "close" to $H_{\max}$, then $f$ and $g$ are "close" in the sense of the $L_1$ norm. Then we select a sequence of functions $f_1, \cdots, f_n, \cdots \varepsilon F$ such that $\lim_{n\to\infty} H(f_n) = H_{\max}$. It follows that $\{f_n\}$ is a Cauchy sequence of functions; by completeness of $L_1$, there is a function $f \varepsilon L_1$ such that $\|f - f_n\|_1 \to 0$. But since $F$ is admissible, it follows that $F$ is $L_1$-closed. Hence, $f \varepsilon F$. We complete proof of the theorem by showing that $H(f) = H_{\max}$. We now proceed to the details of the proof.

LEMMA 2. *Let $F = F(p, q)$ be an admissible set of functions. Then there is a positive constant $A$ depending only on $F$ such that*

$$H(f) \leqq H_+(f) \leqq A \quad \text{for all } f \varepsilon F.$$

PROOF. The first inequality is obvious. The proof of the second is as follows:

$$H_+(f) = \iint_{f<1} f \log \frac{1}{f} = \iint_{f<1} \left[ f(x, y) \log \frac{p(x)q(y)}{f(x, y)} + f(x, y) \log \frac{1}{p(x)q(y)} \right] dx\, dy$$

$$\leqq \iint_{f<1} f(x, y) \left( \frac{p(x)q(y)}{f(x, y)} - 1 \right) dx\, dy + \iint_{f<1} f(x, y) \log \frac{1}{p(x)q(y)}\, dx\, dy$$

$$\leqq \iint p(x)q(y)\, dx\, dy + \iint f(x, y) \left( \left|\log \frac{1}{p(x)}\right| + \left|\log \frac{1}{q(y)}\right| \right) dx\, dy$$

$$\leqq 1 + \int \left| p(x) \log \frac{1}{p(x)} \right| dx + \int \left| q(y) \log \frac{1}{q(y)} \right| dy$$

$$\leqq A,$$

where $A$ depends only on $F$.

An immediate consequence of this lemma is that the following definition makes sense.

DEFINITION 3. We define number $H_{\max}$, $H_+$, and $H_-$ as follows:

$$H_{\max} = \sup_{f \varepsilon F} H(f),$$

$$H_+ = \lim_{\epsilon \to 0} \sup_{f; H(f) \geqq H_{\max} - \epsilon} H_+(f),$$

$$H_- = H_+ - H_{\max}.$$

LEMMA 3. *Let $f$ and $g$ be functions in $F$ such that $H(f)$, $H(g) \geqq H_{\max} - \epsilon$ for some $\epsilon > 0$. Then*

$$\|f - g\|_1 < 9\epsilon^{\frac{1}{2}}.$$

PROOF. Let $\theta f + (1 - \theta)g$, $0 \leqq \theta \leqq 1$, be a convex linear combination of $f$ and $g$. First we show that $\theta f + (1 - \theta)g$ has finite entropy whenever $f$ and $g$ do. Since $\theta f + (1 - \theta)g \varepsilon F$, it follows from Lemma 2 that $H(\theta f + (1 - \theta)g)$ is bounded from above. It remains to show that it is bounded from below. But this is an immediate consequence of Shannon's inequality

$$H(\theta f + (1 - \theta)g) \geqq \theta H(f) + (1 - \theta)H(g), \tag{2}$$

which follows from the concavity of $x \log (1/x)$. Thus the function $h(\theta)$, which we define by the equation $h(\theta) = H(\theta f + (1 - \theta)g)$, is well-defined for all $0 \leqq \theta \leqq 1$. In fact, Definition 3 and Inequality (2) imply that

$$H_{\max} - \epsilon \leqq h(\theta) \leqq H_{\max}. \tag{3}$$

Next we show that $h(\theta)$ has a *second* derivative throughout the open interval $0 < \theta < 1$. A standard argument ([3], p. 323) will be used to prove that $h''$ exists. Thus, let $\theta$ and $\theta' \varepsilon (0, 1)$. We compute (using the mean value theorem)

$$\begin{aligned}\frac{h(\theta)-h(\theta')}{\theta-\theta'} &= \frac{1}{\theta-\theta'}\iint \{[\theta f+(1-\theta)g]\log[(\theta f+(1-\theta)g)^{-1}] \\ &\qquad - [\theta' f+(1-\theta')g]\log[(\theta' f+(1-\theta')g)^{-1}]\} \\ &= \iint [(f-g)\log[(\theta'' f+(1-\theta'')g)^{-1}] - (f-g)] \\ &= \iint (f-g)\log[(\theta'' f+(1-\theta'')g)^{-1}],\end{aligned} \tag{4}$$

where $\theta'' = \theta''(x, y)$ is a measurable function of $(x, y)$ and is between $\theta$ and $\theta'$ for all $(x, y)$. But it is easy to verify that

$$|(f-g)\log[(\theta'' f+(1-\theta'')g)^{-1}]| \leqq |f\log f^{-1}| + |g\log g^{-1}| + f\log[(\theta'')^{-1}] + g\log[(1-\theta'')^{-1}].$$

This inequality shows that

$$(f-g)\log[(\theta'' f+(1-\theta'')g)^{-1}]$$

is bounded by an integrable function. Hence, by Lebesgue's dominated convergence theorem, $h'(\theta)$ exists for $0 < \theta < 1$ and

$$h'(\theta) = \textstyle\iint (f-g)\log[(\theta f+(1-\theta)g)^{-1}]. \tag{5}$$

A similar computation shows that $h''(\theta)$ exists for $0 < \theta < 1$ and that

$$h''(\theta) = -\textstyle\iint (f-g)^2/(\theta f+(1-\theta)g), \qquad 0 < \theta < 1. \tag{6}$$

Now consider the number

$$h = h(\tfrac{2}{3}) - 2h(\tfrac{1}{2}) + h(\tfrac{1}{3}).$$

Inequality (3) implies that $|h| \leqq 2\epsilon$, while the mean value theorem (applied twice) shows that $h = (\frac{1}{36})h''(\theta)$ for some $\frac{1}{3} < \theta < \frac{2}{3}$. Thus, for some $\frac{1}{3} < \theta < \frac{2}{3}$,

$$|h''(\theta)| = \textstyle\iint (f-g)^2/(\theta f+(1-\theta)g) \leqq 72\epsilon.$$

But, by Schwarz's inequality,

$$(\textstyle\iint |f-g|)^2 \leqq \iint (f-g)^2/(\theta f+(1-\theta)g) \iint [\theta f+(1-\theta)g].$$

Therefore,

$$\|f-g\|_1 \leqq (\textstyle\iint (f-g)^2/(\theta f+(1-\theta)g))^{\frac{1}{2}} \leqq (72\epsilon)^{\frac{1}{2}} < 9\epsilon^{\frac{1}{2}},$$

which completes the proof of Lemma 3.

An important consequence of this lemma is that if $f$ and $g$ are two functions in $F$ such that $H(f) = H(g) = H_{\max}$ then $\|f-g\|_1 = 0$; and, therefore, $f = g$ almost everywhere.

We are now in a position to determine the function $f \varepsilon F$ with maximum entropy. To do this, we select a sequence $\{f_n\}$ of functions in $F$ such that

$$\lim_{n\to\infty} H(f_n) = H_{\max}, \tag{7}$$

$$\lim_{n\to\infty} H_+(f_n) = H_+, \tag{8}$$

(9) $$\lim_{n\to\infty} H_-(f_n) = H_- .$$

The existence of such a sequence is guaranteed by Definition 3. Equation (7) and Lemma 3 imply that $f_n$ is a Cauchy sequence ($\|f_n - f_m\|_1 \to 0$ as $n, m \to \infty$). Therefore, by the Reisz-Fisher theorem for $L_1$ (selecting a subsequence of $f_n$ if necessary), there exists a function $f \varepsilon L_1$ such that

(10) $$\lim_{n\to\infty} f_n = f \quad \text{a.e.},$$

and

(11) $$\lim_{n\to\infty} \|f - f_n\|_1 = 0.$$

The last equation implies that $f \varepsilon F$, since $F$ is admissible. Thus we need only show that $H(f) = H_{\max}$ to complete the proof of Theorem 1. That is, we must show

$$\iint f \log f^{-1} = \lim_{n\to\infty} \iint f_n \log f_n^{-1}.$$

Now Equation (10) implies that $f_n \log (1/f_n) \to f \log (1/f)$ a.e. Thus by Fatou's lemma, $f \log (1/f) \varepsilon L_1$ and

(12) $$\iint |f \log f^{-1}| \leqq \lim_{n\to\infty} \iint |f_n \log f_n^{-1}|$$

[$= H_+ + H_-$ by Definitions (8) and (9)]. But, by Definition 3,

(13) $$H(f) = \iint f \log f^{-1} \leqq H_{\max} = H_+ - H_- .$$

Inequalities (12) and (13) are equivalent to the following:

$$H_+(f) + H_-(f) \leqq H_+ + H_- ,$$
$$H_+(f) - H_-(f) \leqq H_+ - H_- .$$

If we can show that $H_+(f) = H_+$, then both inequalities can be replaced by equalities and it will follow from (13) that $H(f) = H_{\max}$ as desired. Thus, to complete the proof of Theorem 1, we need only prove

(14) $$H_+(f) = \lim_{n\to\infty} H_+(f_n).$$

To verify Equation (14) we write

$$\begin{aligned} H_+(f) - H_+(f_n) &= \iint_{f\leqq 1} f \log \frac{1}{f} - \iint_{f_n \leqq 1} f_n \log \frac{1}{f_n} \\ &= \iint_{\substack{f\leqq 1\\ f_n\leqq 1}} f \log \frac{f+f_n}{2f} + \iint_{\substack{f\leqq 1\\ f_n\leqq 1}} (f - f_n) \log \frac{2}{f+f_n} \\ &\quad - \iint_{\substack{f\leqq 1\\ f_n\leqq 1}} f_n \log \frac{f+f_n}{2f_n} \\ &\quad + \iint_{\substack{f\leqq 1\\ f_n> 1}} f \log \frac{1}{f} - \iint_{\substack{f> 1\\ f_n\leqq 1}} f_n \log \frac{1}{f_n} \\ &= I_1 + I_2 - I_3 + I_4 - I_5 \quad \text{say.} \end{aligned}$$

Equation (14) will follow if we show that each of these five terms tends to zero as $n \to \infty$. For the first term we have

$$|f \log [(f + f_n)/2f]| \leqq f \log (f_n/f) \leqq f_n - f \qquad \text{if } f_n \geqq f;$$
$$\leqq f \log [f/(1 - \tfrac{1}{2}(f - f_n))] \leqq 2(f - f_n) \quad \text{if } f \geqq f_n .$$

Thus

$|I_1| \leqq 2\|f - f_n\|_1 \to 0$ as $n \to \infty$. Similarly $|I_3| \to 0$ as $n \to \infty$.

For the second term we have

$$I_2^2 \leqq \left( \iint_{f,f_n \leqq 1} |f - f_n| \log \frac{2}{f + f_n} \right)^2 \leqq \iint_{f,f_n \leqq 1} |f - f_n| \iint_{f,f_n \leqq 1} (f - f_n) \log^2 \frac{2}{f + f_n}$$
$$\leqq \|f - f_n\|_1 \times 2 \times \iint_{f,f_n \leqq 1} \left|\frac{f + f_n}{2}\right| \log^2 \frac{2}{f + f_n} \leqq 2A \, \|f - f_n\|_1 ,$$

by Condition (c) in the definition of admissibility (Section 3). Thus, $I_2 \to 0$ as $n \to \infty$.

For the fourth term we have

$$0 \leqq I_4 = \iint_{\substack{f \leqq 1 \\ f_n > 1}} f \log \frac{1}{f} \leqq \iint_{S_n} f \log \frac{1}{f} ,$$

where

$$S_n = \{(x, y) \mid f(x, y) \leqq 1 \quad \text{and} \quad f_m(x, y) > 1 \quad \text{for } \textit{some} \quad m \geqq n\}.$$

But

$$\lim_{n \to \infty} S_n = \{(x, y) \mid f = 1\} \cup S,$$

where $S$ is a set of measure zero. Thus

$$\lim_{n \to \infty} I_4 \leqq \iint_{\{f=1\} \cup S} f \log \frac{1}{f} = 0.$$

Finally, for the fifth term we have

$$I_5^2 = \left(\iint_{\substack{f_n \leqq 1 \\ f > 1}} f_n \log \frac{1}{f_n}\right)^2 \leqq \iint_{\substack{f_n \leqq 1 \\ f > 1}} f_n \iint f_n \log^2 \frac{1}{f_n} \leqq A \iint_{\substack{f_n \leqq 1 \\ f > 1}} f_n$$
$$\leqq A \iint_{\substack{f_n \leqq 1 \\ f > 1}} |f_n - f| + A \iint_{\substack{f_n \leqq 1 \\ f > 1}} f \leqq A \, \|f_n - f\|_1 + A \iint_{T_n} F,$$

where

$$T_n = \{(x, y) \mid f(x, y) > 1 \quad \text{and} \quad f_m(x, y) \leqq 1 \quad \text{for some} \quad m \geqq n\}.$$

But $\lim_{n\to\infty} T_n$ is a set of measure zero by Equation (10). Thus $I_5 \to 0$ as $n \to \infty$, which completes the proof of Equation (14), and thus of Theorem 1.

**5. Characterization of solution.** The following theorem characterizes the function with maximum entropy, revealing its special form.

THEOREM 2. *Let* $F = F(p, q; r_1 \cdots, r_k; \rho_1, \cdots, \rho_k)$ *be an admissible set and let* $f$ *be the unique function in* $F$ *with maximum entropy. The* $f$ *has the form*

$$f(x, y) = a(x)b(y) \exp\left[\textstyle\sum_{j=1}^{k} \alpha_j r_j(x, y)\right], \tag{15}$$

*where* $a(x)$ *and* $b(y)$ *are positive functions of* $x$ *and* $y$, *respectively and* $\alpha_1, \cdots, \alpha_k$ *are constants. Furthermore,* $f$ *is uniquely characterized by Equation* (15); *that is,* $f$ *is the only element of* $F$ *(up to almost everywhere equivalence) which has this form,* $a(x)$ *and* $b(y)$ *are unique to almost everywhere equivalence, and the* $\alpha_j$ *are unique constants.*

REMARK. Equation (15) can be obtained formally by using a calculus of variations type of argument to maximize the integral $\iint f \log f^{-1}$, where $f$ is subject to the constraints

$$\textstyle\int_y f = p, \qquad \int_x f = q, \qquad \iint f r_j = \rho_j .$$

The following proof should be looked upon as the rigorous justification of this purely formal calculus of variations analysis. The referee pointed out that the uniqueness part of the proof of Theorem 2 can also be done by a clever application of Jensen's inequality (the expected value of log $u(x, y)$ is less than the log of the expected value, unless $u$ is constant almost everywhere) in special cases.

PROOF OF THEOREM 2. Let $g \varepsilon F$ be such that $g < Af$ a.e. for some constant $A$. We first show that

$$\iint (g - f) \log f^{-1} = 0. \tag{16}$$

As before, we define the function $h(\theta)$ by the equation $h(\theta) = H[\theta g + (1 - \theta)f]$. Since $g \leqq Af$ a.e., the function $h(\theta)$ is defined in an interval $[\delta, 1]$ for some $\delta > 0$. From the definition of $f$, it is clear that $h(\theta)$ has a maximum value at $\theta = 0$. But an analysis like that used in the proof of Lemma 3 shows that $h''(0)$ exists ($g \leqq Af$ is not really necessary here) and equals $\iint (g - f) \log (1/f)$. But $h'(0) = 0$, and therefore Equation (16) holds.

Now let $B$ be the set defined by

$$B = L_1(f) = \{\omega = \omega(x, y) \mid \iint |\omega| f < \infty\}.$$

It is well known ([4], Section 15) that $B$ is a Banach space and that its dual space $B^*$ is given by

$$B^* = L_\infty(f) = \{\mu \mid |\mu| \leqq Af \quad \text{a.e.,} \qquad \text{for some constant } A\}.$$

Let $S$ be the subset of $B$ defined by

$$S = \{\omega \varepsilon B \mid \omega = \alpha(x) + \beta(y) + \textstyle\sum_{j=1}^{k} \alpha_j r_j(x, y)\}$$

for some functions $\alpha$ and $\beta$, and some constant $\alpha_1, \cdots, \alpha_k$. Observe that $S$

is a closed subspace of $B$. Then let $S^{\perp}$ be defined as the orthogonal complement of $S$ in $B^*$:

$$S^{\perp} = \{\mu \in B^* \mid \iint \mu\omega = 0 \quad \text{for all} \quad \omega \in S\}.$$

A simple argument shows that $S^{\perp}$ can also be written as the following set of functions:

$$S^{\perp} = \{\mu \mid |\mu| \leqq Af \quad \text{a.e.,} \qquad \text{for some constant } A \tag{17}$$

and

$$\iint \mu r_j = \int_x \mu = \int_y \mu = 0\}.$$

Write $\Phi = \log(1/f)$ and observe that $\Phi \in B$, since $H(f)$ exists. Equations (16) and (17) imply that $\Phi$ is orthogonal to $S^{\perp}$ (i.e., that $\iint \Phi\mu = 0$ whenever $\mu \in S^{\perp}$), since if $e \in S^{\perp}$, then $e + f \in F$ by Equation (17). But if $S$ is a closed subspace of a Banach space $B$ then $(S^{\perp})^{\perp} \cap B = S$, ([4], p. 20). It follows that $\log(1/f) = \Phi$ is in $S$. That is,

$$\log f^{-1} = \alpha(x) + \beta(y) + \textstyle\sum_{j=1}^{k} \alpha_j r_j(x, y),$$

the required expression. This proved the first part of Theorem 2. To prove that $f$ is in fact uniquely characterized by the fact that it belongs to $F$ and has the form $a(x)b(y) \exp[\sum \alpha_j r_j(x, y)]$, let $g$ be another member of $F$ which has this same form. Then we observe that $f \log g$ and $g \log f$ are in $L_1$ (this follows immediately from the special forms of $f$ and $g$ and the fact that $fr_j$ and $gr_j$ are in $L_1$ for $j = 1, \cdots, k$). But then a simple extension of the argument used in the proof of Lemma 3 shows that the function $h(\theta) = H[\theta f + (1 - \theta)g]$ has a (right-handed) derivative at $\theta = 0$ and a (left-handed) derivative at $\theta = 1$. In fact, we obtain

$$h'(0) = \iint (g - f) \log g$$
$$h'(1) = \iint (g - f) \log f.$$

But both of these integrals are equal to zero:

$$\iint (g - f) \log f = \iint (g - f) \log a(x) + \iint (g - f) \log b(y) + \iint (g - f) \sum \alpha_j r_j = 0.$$

[The first two integrals on the right are zero because $\int (g - f)\, dy = \int (g - f)\, dx = 0$; the third is zero because each $(g - f)r_j$ is zero.] Therefore, by Rolle's Theorem, there is *some* $0 < \theta < 1$ such that $h''(\theta) = 0$. And by Schwarz's inequality

$$\|f - g\|_1 \leqq \left(\iint \frac{(f - g)^2}{\theta f + (1 - \theta)g}\right)^{\frac{1}{2}} \left(\iint \theta f + (1 - \theta)g\right)^{\frac{1}{2}} = |h''(\theta)|^{\frac{1}{2}}$$

for all $\theta$ with $0 < \theta < 1$. Let $\theta$ be such that $h''(\theta) = 0$. Then it can be concluded that $\|f - g\|_1 = 0$, and $f = g$ a.e. To prove the uniqueness of $a(x)$, $b(y)$, and the $\alpha_j$, observe that if $f(x, y)$ were expressed in two different ways almost everywhere in the form given by Equation (15), then some nonzero linear combination of the $r_j(x, y)$ would equal a function of $x$ plus a function of $y$ almost everywhere. But this possibility is not permitted by the definitions of Section 3. This completes the proof of Theorem 2.

**6. Main theorem.** We restate and summarize Theorems 1 and 2 in the following theorem.

THEOREM 3. *Let the functions $p(x)$, $q(y)$, $r_1(x, y), \cdots, r_k(x, y)$ and the constants $\rho_1, \cdots, \rho_k$ be such that $F = F(p, q; r_1, \cdots, r_k; \rho_1, \cdots, \rho_k)$ is an admissible set. Then the set of simultaneous integral equations*

$$a(x)\int_{-\infty}^{\infty} b(y) \exp\left[\sum \alpha_j r_j(x, y)\right] dy = p(x);$$

$$b(y)\int_{-\infty}^{\infty} a(x) \exp\left[\sum \alpha_j r_j(x, y)\right] dx = q(y);$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{a(x)b(y) \exp\left[\sum \alpha_j r_j(x, y)\right]\} r_i(x, y)\, dx\, dy = \rho_i, \quad i = 1, \cdots, k,$$

*has a unique solution in functions $a(x)$, $b(y)$ and constants $\alpha_1, \cdots, \alpha_k$. Furthermore the function $a(x)b(y)\exp(\sum \alpha_j r_j)$ is the unique element of $F$ with maximum entropy.*

REMARK. Note that when $k = 0$ (no constraints), then

$$\sum_{i=1}^{n} \alpha_i r_i(x, y) = 0$$

and $f(x, y) = a(x)b(y)$. The unique solution (up to constant factors) is, of course, $a(x) = p(x)$, $b(y) = q(y)$. This is nothing but the well-known result of Shannon ([6], p. 17) that the joint distribution of maximum entropy with prescribed marginals is the product of the marginal distributions, obtained when $X$ and $Y$ are independent random variables.

**7. A particular symmetric case.** In the case of the lunar radar problem discussed in Section 1, we can make the following observations. From [5], in suitable height units,

$$p(x) = q(x) = \tfrac{1}{2}e^{-|x|}; \qquad k = 1; \qquad r_1(x, y) = |x - y|.$$

The expected value $\rho$ of $r_1(x, y)$ is assumed known.

It is easy to prove from the concavity of the entropy functional $H(f)$ that, in case $p(x) = q(x)$, and all $r_j(x, y)$ are symmetric in $x$ and $y$ (as is $r_1(x, y) = |x - y|$), then the $f(x, y)$ providing maximum entropy is likewise symmetric. That is, $f(x, y) = f(y, x)$ when $p = q$ and each $r_j$ is symmetric. Therefore, the set of integral equations of Theorem 3 for this case becomes

$$b(x)\int_{-\infty}^{\infty} b(y)e^{\alpha|x-y|}\, dy = \tfrac{1}{2}e^{-|x|},$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x - y| b(x)b(y)e^{\alpha|x-y|}\, dx\, dy = \rho.$$

We would have an admissible family for this problem by the remarks following Lemma 1 in Section 3 ($x^4 e^{-|x|} \varepsilon L_1$) provided we can show that the family $F$ is nonempty for every possible $\rho$. We shall observe below that $0 < \rho < 2$. To actually exhibit an element of $F$ for every such $\rho$ is easy enough to do in several different ways. One way is to obtain distributions (which are not densities) corresponding to $\rho = 0$ and 2; we shall obtain these two distributions below. Perturb these slightly to obtain densities with $\rho = \epsilon, 2 - \epsilon$, for any $\epsilon$, $0 < \epsilon < 1$. Take convex linear combinations of these two densities to obtain an element of $F$ with any $\rho$ between $\epsilon$ and $2 - \epsilon$. Since $\epsilon$ is arbitrary in $(0, 1)$, any $\rho$ in $(0, 2)$ is so obtainable. The perturbation process needed to obtain densities from distributions simply amounts to replacing a delta function conditional distribution with an approximate identity in $L_1$. Further details are omitted. We remark that if the integral equation were found to have a positive solution for a given $\rho$, a demonstration that $F$ is nonempty would be automatically obtained. Now that $F$ has been shown to be nonempty, we conclude that the family $F$ is admissible in the sense of Section 3. Hence, the theorems of this paper now can be applied.

The way these integral equations would be used is as follows. "Solve" the first integral equation for $b(x)$ *as a function of the parameter* $\alpha$. To obtain the solution is however a formidable assignment; we have not yet been able to solve this problem. The solution would yield an $f_\alpha(x, y)$ say as the density function sought. Then find $E_\alpha(|X - Y|)$, where $E_\alpha$ denotes $\iint \cdots f_\alpha(x, y)\, dx\, dy$. The function $E_\alpha(|X - Y|)$ of $\alpha$ is set equal to $\rho$. This determines $\alpha$ as a function of $\rho$ and thus $\rho$ as a function of $\alpha$. The value of $\alpha$ so determined is inserted into $f_\alpha(x, y) = b(x)b(y)e^{\alpha|x-y|}$; this $f_\alpha$ is the unique $f(x, y)$ of maximum entropy for the given value of $\rho$.

A problem that such a solution method would encounter is this: Is $\alpha$ determined uniquely from $\rho$? But Theorem 1 gives the answer: yes. For a function of the form $b(x)b(y)e^{\alpha|x-y|}$ is of this form for exactly one $\alpha$, as was proved in Theorem 2. Since $f(x, y)$ is uniquely determined by $\rho$ (Theorem 1), $\rho$ conversely uniquely determines $\alpha$. Thus, as $\rho$ increases from 0, $\alpha$ goes from some minimum value; one obtains a one-to-one function. (We are indebted to A. Garsia for a proof that the mapping from $\rho_j$-space into $\alpha_j$-space is always one-to-one.) Now $\rho = 0$ when (and only when) $X$ and $Y$ are proportional (with probability 1). (The "density" function becomes $f(x, y) = \frac{1}{2} \exp\{[(-|x| - |y|)/2]\delta(x - y)\}$, $\delta$ is the Dirac delta function). This value $\rho = 0$ clearly corresponds to $\alpha = -\infty$. Thus, $\alpha$ is a one-to-one function of $\rho$ in some set of $\rho$. This set of $\rho$ is an interval by our earlier remarks in this section.

What is the largest value of $\alpha$ that can occur? And what is the largest value of $\rho$ that can occur? Since $E(|X - Y|) < E(|X|) + E(|Y|) = 2$, $\rho = 2$ is the largest $\rho$ to consider. Moreover, $E(|X - Y|)$ is strictly less than $E(|X|) + E(|Y|)$, unless $X + Y = 0$ a.e. Thus $\rho = 2$ occurs only for $X = -Y$ a.e., corresponding to $f(x, y) = \frac{1}{2} \exp\{[(-|x| - |y|)/2]\delta(x + y)\}$; this case corresponds to $\alpha = +\infty$. Thus, we need consider only the open interval $0 < \rho < 2$ when solving the integral equation for $b(x)$.

This case $\rho = 2$ does not contradict Theorem 3, since the set $F$ for $\rho = 2$ is not admissible in the sense of Section 3. A similar comment applies to the case

$\rho = 0$. Actually, the case $\rho = 2$ can be thought of as the limit of solutions of the form $\exp [a(|x - y| - |x| - |y|)]b(x)b(y)$, as $a \to \infty$. For then only the case $x = -y$ gives a term in the limit; $b(x)$ must then approach $2^{-\frac{1}{2}} \exp(-|x|)/2$.

In sum, as $\rho$ goes from 0 to 2, $\alpha$ goes from $-\infty$ to $+\infty$, and $\alpha$ is a one-to-one function of $\rho$. We can invert this function in one trivial case: when $\alpha = 0$, $X$ and $Y$ are independent, and $f(x, y) = \frac{1}{4} \exp(-|x| - |y|)$. The value of $\rho$ is then $\frac{3}{2}$. Thus $\rho = \frac{3}{2}$ corresponds to $\alpha = 0$.

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