# THE DISTRIBUTION OF THE GENERALIZED VARIANCE

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**1. Introduction.** The generalized variance i.e. the determinant of the sample variance and covariance matrix is defined [10] to be a measure of the spread of observations. Let $S$ be the sample variance and covariance matrix of order $(p \times p)$ with $n_1$ degrees of freedom (d.f.) and let $\Sigma(p \times p) = E(n_1 S)$. The $h$th moment of the det. $|A| (= |n_1 S|)$, in the central case is given by Wilks [10] and that, in the noncentral case, by Herz [8] in the form of Laguerre polynomials and also by Constantine [7] in the form of Gaussian hypergeometric function of the type ${}_1F_1$.

Let $k_i^2$ $(i = 1, 2, \cdots, p)$ be the real and non-negative roots of the determinantal equation

$$|T - k^2 \Sigma| = 0$$

where $T$ is the noncentrality matrix of $S$. Assuming $k_i^2 = 0$ $(i = 2, 3, \cdots, p)$ and $k_1^2 \neq 0$, Anderson [1] gives the $h$th moment of the det. $|A|$ in the noncentral linear case as follows:

$$E(|A|^h) = 2^{ph} \exp\left(-\tfrac{1}{2} k_1^2\right) \prod_{i=1}^{p-1} \frac{\Gamma[\frac{1}{2}(n_1 - i) + h]}{\Gamma[\frac{1}{2}(n_1 - i)]} \cdot \sum_{j=0}^{\infty} \left\{ \frac{k_1^{2j}}{2^j j!} \frac{\Gamma(\frac{1}{2} n_1 + j + h)}{\Gamma(\frac{1}{2} n_1 + j)} \right\}. \tag{1.1}$$

It has been found difficult to obtain the distribution of the det $|A|$ in the noncentral case by making use either of the $h$th moments given by Herz [8] or that of Constantine [7]. The determination of the distribution of the det. $|A|$ by taking its $h$th moment (as in (1.1)), in the noncentral linear case for various values of $p$, has been found easy. For $p = 2$, 3 and 4 the author [3], [5], has already determined the distribution of the det. $|A|$ both for central and noncentral linear cases. We list only their results for completeness. In Section 3 the distribution of the det. $|A|$ in the noncentral linear case for higher values of $p = 5(1)10$ has been found and put in the standard form of the generalized Gauss' hypergeometric series defined as

$$ {}_0F_t(; r_1, r_2, \cdots, r_t; a) = 1 + \frac{1}{r_1 r_2 \cdots r_t} \frac{a}{1!} + \frac{1}{r_1(r_1+1)\, r_2(r_2+1) \cdots r_t(r_t+1)} \frac{a^2}{2!} + \cdots. \tag{1.2}$$

Then to determine the distribution in the central case for the same values of $p$,

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the noncentrality parameter $k_1^2$ is set equal to zero in each of the distributions found for the noncentral case. The method is so general and straightforward that it can be easily extended to higher values of $p$ also.

**2. Some preliminaries and definite integrals.** We list below Legendre's duplication formula and the values of some definite integrals either taken from standard books of tables on integrals or evaluated by the author himself by following the evaluation procedure discussed in his paper [4].

(i) Legendre's duplication formula for the gamma function

$$\Gamma(n + \tfrac{1}{2})\Gamma(n + 1) = \pi^{\frac{1}{2}}\Gamma(2n + 1)/2^{2n}. \tag{2.1}$$

(ii) For $a \geqq 0$, the Larsen's tables [9] gives

$$\int_0^\infty \exp(-x^2 - a^2x^{-2})\,dx = \tfrac{1}{2}\pi^{\frac{1}{2}} \exp(-2a). \tag{2.2}$$

(iii) Bierens deHann ([6], pp. 143–144) gives in his Table 98 the following two integrals:

$$\begin{aligned}\int_0^\infty & x^{a-\frac{1}{2}} \exp(-Px - Qx^{-1})\,dx \\ &= \left(\frac{Q}{P}\right)^{\frac{1}{2}a} \exp[-2(PQ)^{\frac{1}{2}}] \left(\frac{\pi}{P}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left[\frac{(a+1-n)^{2n/1}}{2^{n/2}(2(PQ)^{\frac{1}{2}})^n}\right]\end{aligned} \tag{2.3}$$

$$\begin{aligned}\int_0^\infty & x^{-a-\frac{1}{2}} \exp(-Px-Qx^{-1})\,dx \\ &= \left(\frac{P}{Q}\right)^{\frac{1}{2}a} \exp[-2(PQ)^{\frac{1}{2}}] \left(\frac{\pi}{P}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left[\frac{(a-n)^{2n/1}}{2^{n/2}(2(PQ)^{\frac{1}{2}})^n}\right].\end{aligned} \tag{2.4}$$

(In both of these, Kramp's notation is used, namely

$$x^{n/h} \equiv x(x+h)(x+2h)\cdots(x+\overline{n-1}\,h).)$$

(iv) We finally list the following four integrals:

$$D_r(a) = \int_0^\infty x^{\frac{1}{2}r} \exp(-ax^{-\frac{1}{2}} - x)\,dx, \tag{2.5}$$

$$K_r(a) = \int_0^\infty x^{\frac{1}{2}r+1} \exp(-ax^{-\frac{1}{2}} - x^2)\,dx, \tag{2.6}$$

$$Q_r(a) = \int_0^\infty x^{\frac{1}{2}r+3} \exp(-ax^{-\frac{1}{2}} - x^4)\,dx, \tag{2.7}$$

and

$$L_r(a) = 2\int_0^\infty x^{2r+1} \exp(-ax^{-1} - x^2)\,dx, \tag{2.8}$$

where $a$ in each case is real and positive.

We have evaluated them by the method already discussed by the author in his paper [4] and give the value of each for some suitably chosen of $r$. They are:

$$\begin{aligned}D_{9/2}(a) &= \Gamma(13/4)\,{}_0F_2(;\,-9/4,\,1/2; \\ &\quad -a^2/4) - a\Gamma(11/4)\,{}_0F_2(;\,-7/4,\,3/2;\,-a^2/4)\end{aligned} \tag{2.9}$$

$$- 2\pi^{\frac{1}{2}}[4a)^{13/2}/13!]6!\ {}_0F_2(;15/4, 17/4; -a^2/4),$$

$$K_{5/2}(a) = \tfrac{1}{2}\Gamma(13/8)\ {}_0F_4(; -5/8, 1/4, 1/2, 3/4, -a^4/256)$$

$$- (a/1!)\tfrac{1}{2}\Gamma(11/8)\ {}_0F_4(; -3/8, 1/2, 3/4, 5/4; -a^4/256)$$

$$+ (a^2/2!)\tfrac{1}{2}\Gamma(9/8)\ {}_0F_4(; -1/8, 3/4, 5/4, 3/2; -a^4/256) \tag{2.10}$$

$$- (a^3/3!)\tfrac{1}{2}\Gamma(7/8)\ {}_0F_4(; 1/8, 5/4, 3/2, 7/4; -a^4/256)$$

$$- 2\pi^{\frac{1}{2}}[(4a)^{13/2}/13!]6!\ {}_0F_4(;15/8, 17/8, 19/8, 21/8; -a^4/256),$$

$$Q_{-1/2}(a) = \tfrac{1}{4}\Gamma(15/16)\ {}_0F_8(; 1/16, 1/8, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8; -a^8/(256)^3)$$

$$- (a/1!)\tfrac{1}{4}\Gamma(13/16)\ {}_0F_8(; 3/16, 1/4, 3/8, 1/2, 5/8, 3/4, 7/8, 9/8; -a^8/(256)^3)$$

$$+ (a^2/2!)\tfrac{1}{4}\Gamma(11/16)\ {}_0F_8(; 5/16, 3/8, 1/2, 5/8, 3/4, 7/8, 9/8, 5/4; -a^8/(256)^3)$$

$$- (a^3/3!)\tfrac{1}{4}\Gamma(9/16)\ {}_0F_8(; 7/16, 1/2, 5/8, 3/4, 7/8, 9/8, 5/4, 11/8; -a^8/(256)^3)$$

$$+ (a^4/4!)\tfrac{1}{4}\Gamma(7/16)\ {}_0F_8(; 9/16, 5/8, 3/4, 7/8, 9/8, 5/4, 11/8, 3/2; -a^8/(256)^3) \tag{2.11}$$

$$- (a^5/5!)\tfrac{1}{4}\Gamma(5/16)\ {}_0F_8(; 11/16, 3/4, 7/8, 9/8, 5/4, 11/8, 3/2, 13/8; -a^8/(256)^3)$$

$$+ (a^6/6!)\tfrac{1}{4}\Gamma(3/16)\ {}_0F_8(; 13/16, 7/8, 9/8, 5/4, 11/8, 3/2, 13/8, 7/4; -a^8/(256)^3)$$

$$- (a^7/7!)\tfrac{1}{4}\Gamma(1/16)\ {}_0F_8(; 15/16, 9/8, 5/4, 11/8, 3/2, 13/8, 7/4, 15/8; -a^8/(256)^3)$$

$$+ 2\pi^{\frac{1}{2}}[(4a)^{15/2}/15!]7!\ {}_0F_8(; 17/16, 19/16, 21/16, 23/16, 25/16, 27/16, 29/16, 31/16; -a^8/(256)^3),$$

and

$$L_0(a) = [1 + 2(a^2/2!)(\tfrac{1}{2} + 1) - (2^2/2)(a^4/4!)(\tfrac{1}{4} + \tfrac{1}{3} + \tfrac{1}{2} + 1 + \tfrac{1}{2})$$

$$+ (2^3/2\cdot 4)(a^6/6!)(\tfrac{1}{6} + \tfrac{1}{5} + \cdots + \tfrac{1}{2} + 1 + \tfrac{1}{2} + \tfrac{1}{4}) - \cdots]$$

$$- a\pi^{\frac{1}{2}}\ {}_0F_2(; \tfrac{1}{2}, \tfrac{3}{2}; -a^2/4) \tag{2.12}$$

$$- 2(\gamma + \log a)(a^2/2!)\ {}_0F_2(; \tfrac{3}{2}, 2; -a^2/4),$$

where $\gamma$ is the Euler's constant and ${}_0F_t$ are as defined in (1.2).

REMARK A. In each of the above four integrals we have given its value for some specified value of $r$. In order to find the value of the same integral for both positive and negative values less than the specified value of $r$, we differentiate with respect to $a$ successively. For values of $r$ larger than the specified value we integrate the integral with respect to $a$ and evaluate the constant of integration by setting $a = 0$ on both sides.

**3. Distributions.** We [2] have already proved with the help of (1.1) that the distribution of the det. $|A|$ in the noncentral linear case is the same as that of the product $u_0 u_1 \cdots u_{p-1}$ where the joint distribution of $u_i$ is the following:

$$(3\text{-A})\quad 2^{-p[m+(p+3)/4]} \prod_{i=1}^{p-1} u_{p-i-1}^{m+i/2} \left[\Gamma\left(m + \frac{i}{2} + 1\right)\right]^{-1} \exp\left(-\frac{1}{2}\sum_{i=0}^{p-1} u_i - \frac{1}{2}k_1^2\right) \times \left[1 + \frac{u_0}{1!}\frac{k_1^2}{2(2m+p+1)} + \frac{u_0^2}{2!}\frac{k_1^4}{4(2m+p+1)(2m+p+3)} + \cdots\right] \cdot \prod_{i=0}^{p-1} du_i,$$

where $0 \leqq u_0, u_1, \cdots, u_{p-1} \leqq \infty$ and $n_1 = 2m + p + 1$, $(p \leqq m)$. We substitute successively the various values of $p$ in (3-A) and get the distribution of the products in $u$'s i.e. of the det. $|A|$ of various orders. Since the distributions of the det. $|A|$ of orders 2, 3 and 4 are already known [3], [5], we simply list below their results. For the higher order determinants, we actually determine their distributions.

(i) *For* $p = 2$. The distribution, for $p = 2$ in the noncentral linear case, of $V_1 = |A|^{\frac{1}{2}}$ is

$$(3.1)\quad V_1^{2m+1} \exp(-V_1 - \tfrac{1}{2}k_1^2)[\Gamma(2m+1)]^{-1}[1 + (T_1/1!)[k_1^2/(2m+3)] + (T_2/2!)[k_1^4/((2m+3)(2m+5))] + \cdots] \quad (0 \leqq V_1 \leqq \infty)$$

where $m = \frac{1}{2}(n_1 - 3)$ and $T_r(r \neq 0)$ is defined as

$$T_r = (\tfrac{1}{2}V_1)^r \textstyle\sum_{n=0}^{\infty} [(r + 1 - n)^{2n/1}/2^{n/2} V_1^n],$$

again using Kramp's notation.

It follows from (3.1) that in the central case i.e. when $k_1^2 = 0$ and $m = \frac{1}{2}(n_1 - 3)$, the distribution of $V_1(=|A|^{\frac{1}{2}})$ is

$$(3.2)\quad [\Gamma(n_1 - 1)]^{-1} V_1^{n_1-2} \exp(-V_1)\, dV_1, \qquad (0 \leqq V_1 \leqq \infty),$$

which is a gamma variate with parameter $(n_1 - 1)$.

(ii) *For* $p = 3$. The distribution for $p = 3$ in the noncentral linear case, of $V_1(=|A|^{\frac{1}{2}})$ is

$$(3.3)\quad 2^{-m-1}[\Gamma(m+1)\Gamma(2m+3)]^{-1} V_1^m \exp(-\tfrac{1}{2}k_1^2) \left[L_0((\tfrac{1}{2}V_1)^{\frac{1}{2}}) + \frac{k_1^2}{1!\,(2m+4)} L_1((\tfrac{1}{2}V_1)^{\frac{1}{2}}) + \frac{k_1^4}{2!\,(2m+4)(2m+6)} L_2(\tfrac{1}{2}V_1)^{\frac{1}{2}}) + \cdots\right] dV_1$$

where $0 \leqq V_1 \leqq \infty$, $m = \frac{1}{2}(n_1 - 4)$ and $L$'s are defined as in (2.8), (2.12) and Remark A.

The distribution, in the central case, is obvious when we set $k_1^2 = 0$ and $m = \frac{1}{2}(n_1 - 4)$ in (3.3).

(iii) *For* $p = 4$. The distribution for $p = 4$ of $V_1(=|A|)$ in the noncentral linear case is

$$\frac{1}{2}[\Gamma(2m+2)\Gamma(2m+4)]^{-1}V_1^m \exp(-\tfrac{1}{2}k_1^2) \int_0^\infty V_3 \cdot \exp(-V_3^{-1}(V_1)^{\frac{1}{2}} - V_3) \times \left[1 + \frac{I_1}{1!}\frac{k_1^2}{2m+5} + \frac{I_2}{2!}\frac{k_1^4}{(2m+5)(2m+7)} + \cdots\right] dV_3\, dV_1 \qquad (0 \leqq V_1 \leqq \infty) \tag{3.4}$$

where $I_r$ is defined as follows:

$$I_r = \tfrac{1}{2}\pi^{\frac{1}{2}}(\tfrac{1}{2}V_3)^r \exp(-V_3) \textstyle\sum_{n=0}^\infty [(r+1-n)^{2n/1}/(2^{n/2}V_3^n)].$$

Further, to evaluate (3.4), we need to use (2.3) and (2.4), as the need be, for $P = 1$ and $Q = (V_1)^{\frac{1}{2}}$ and various suitable values of $a$. This determines the distribution of $V_1(=|A|)$ in the noncentral linear case for $p = 4$ where $m = \frac{1}{2}(n_1 - 5)$. For the central case i.e. when $k_1^2 = 0$, the distribution of $V_1(=|A|)$ is

$$\tfrac{1}{2}\pi^{\frac{1}{2}}[\Gamma(2m+2)\Gamma(2m+4)]^{-1} \cdot \textstyle\sum_{n=0}^\infty [A_n V_1^{m+(3/8)-(n/4)}] \exp(-2V_1^{\frac{1}{4}})\, dV_1 \quad (0 \leqq V_1 \leqq \infty) \tag{3.5}$$

where

$$A_n = [(5/2) - n]^{2n/1}/(2^{n/2}\cdot 2^n). \tag{3.6}$$

(iv) *For* $p = 5$. Setting $p = 5$ and $u_0u_1u_2u_3u_4 = V_1$, $u_0u_1u_2u_3 = V_2^2$, $u_0u_1u_2 = V_3^2$, $u_0u_1 = V_4^2$ and $u_0 = V_5^2$ in (3-A) and then applying (2.1) the distribution of $V_1(=|A|)$ is

$$V_1^m \exp(-\tfrac{1}{2}k_1^2)\left[2^{m+2}\pi\Gamma(m+3)\prod_{i=1}^{2}\Gamma(2m+2i)\right]^{-1} \cdot \int_0^\infty \cdots \int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{V_1}{V_2^2} + \frac{V_2^2}{V_3^2} + \frac{V_3^2}{V_4^2} + \frac{V_4^2}{V_5^2} + V_5^2\right)\right] dV_1 \times \left[1 + \frac{V_5^2}{1!}\frac{k_1^2}{2(2m+6)} + \frac{V_5^4}{2!}\frac{k_1^4}{4(2m+6)(2m+8)} + \cdots\right]\prod_{i=2}^{5} dV_i\,,$$

where $(0 \leqq V_1 \leqq \infty)$. We evaluate it first for $V_2$ and $V_4$ with the help of (2.2) and then for $V_3$ with the help of (2.3). Finally we set $V_5 = 2^{\frac{1}{2}}t$ and after using (2.6) we get

$$(\pi)^{\frac{1}{2}}V_1^{m+(3/8)} \exp(-\tfrac{1}{2}k_1^2)\left[2^{m+(9/8)}\Gamma(m+3)\prod_{i=1}^{2}\Gamma(2m+2i)\right]^{-1}$$

$$(3.7) \qquad \times \left[ \sum_{n=0}^{\infty} (2^{n/4} A_n V_1^{-n/4} K_{n+\frac{5}{2}}) + \frac{k_1^2}{2m+6} \sum_{n=0}^{\infty} (2^{n/4} A_n V_1^{-n/4} K_{n+(13/2)} + \frac{k_1^4}{2!\,(2m+6)(2m+8)} \sum_{n=0}^{\infty} (2^{n/4} A_n V_1^{-n/4} K_{n+(21/2)}) + \cdots \right] dV_1,$$

$$(0 \leqq V_1 \leqq \infty),$$

where $A_n$ is as in (3.6) and $K$'s are as in (2.6), (2.10) and Remark A for $a = (8V_1)^{\frac{1}{4}}$. This determines the distribution of $V_1(=u_0 u_1 \cdots u_4)$ i.e. of the determinant $|A|$ of order 5 where it is to be noted that $m = \frac{1}{2}(n_1 - 6)$.

For the central case, we set $k_1^2 = 0$ in (3.7). We get

$$(3.8) \qquad (\pi)^{\frac{1}{2}} V_1^{m+(3/8)} [2^{m+(9/8)} \Gamma(m+3) \cdot \textstyle\prod_{i=1}^{2} \Gamma(2m+2i)]^{-1} \sum_{n=0}^{\infty} [2^{n/4} V_1^{-n/4} A_n K_{n+(5/2)}]\, dV_1,$$

where $0 \leqq V_1 \leqq \infty$.

(v) *For* $p = 6$. Setting $p = 6$ in (3-A) and then making the usual substitutions for $u$'s in terms of $V$'s, as done in the case (iv) for $p = 5$, the distribution of $V_1(=u_0 u_1 \cdots u_5)$ after the use of (2.1) is

$$2^{\frac{1}{2}} V_1^m \exp(-\tfrac{1}{2}k_1^2)\, dV_1 \left[ \pi^{\frac{3}{2}} \prod_{i=1}^{2} \Gamma(2m+2i) \right]^{-1} \times \int_0^{\infty} \cdots \int_0^{\infty} \exp\left[ -\frac{1}{2} \left( \frac{V_1}{V_2^2} + \frac{V_2^2}{V_3^2} + \cdots + \frac{V_5^2}{V_6^2} + V_6^2 \right) \right] \times \left[ 1 + \frac{V_6^2}{1!} \frac{k_1^2}{2(2m+7)} + \frac{V_6^4}{2!} \frac{k_1^4}{4(2m+7)(2m+9)} + \cdots \right] \prod_{i=2}^{6} dV_i,$$

where $(0 \leqq V_1 \leqq \infty)$. We evaluate it first for $V_2$ and $V_4$ by the use of (2.2) and then for $V_3$ with the help of (2.3). Finally the evaluation is made with respect to $V_3$ by making use of both (2.2) and (2.3), so that the distribution of $V_1$ is

$$\tfrac{1}{2}(\pi)^{\frac{1}{2}} V_1^{m+(3/8)} \exp(-\tfrac{1}{2}k_1^2) \left[ \prod_{i=1}^{3} \Gamma(2m+2i) \right]^{-1} dV_1$$

$$(3.9) \qquad \times \int_0^{\infty} \left[ \sum_{n=0}^{\infty} A_n V_5^{(n/2)+(9/4)} V_1^{-n/4} \right] \exp(-2V_1^{\frac{1}{4}} V_5^{-\frac{1}{2}} - V_5) \times \left[ 1 + \frac{T_1'}{1!} \frac{k_1^2}{2(2m+7)} + \frac{T_2'}{2!} \frac{k_1^4}{4(2m+7)(2m+9)} + \cdots \right] dV_5$$

where $(0 \leqq V_1 \leqq \infty)$,

$$(3.10) \qquad T_r' = \sum_{n=0}^{\infty} (A_n' V_5^{r-n})$$

and

$$(3.11) \qquad A_n' = (r+1-n)^{2n/1}/2^{n/2}.$$

(Kramp's notation is again used, i.e., $x^{n/h} = x(x+h)(x+2h) \cdots$

$(x + \overline{n - 1}h)$.) Now to evaluate (3.16) we use (2.5), (2.9) and Remark A and and then get the distribution of $V_1(=u_0u_1u_2 \cdots u_5)$ i.e. of the determinant $|A|$ of order 6 in the noncentral linear case where it may be noted that $m = \frac{1}{2}(n_1 - 7)$.

For the central case, we set $k_1^2 = 0$ in (2.9) and get

$$(3.12)\quad \tfrac{1}{2}(\pi)^{\frac{1}{2}}V_1^{m+(3/8)}[\textstyle\prod_{i=1}^{3}\Gamma(2m + 2i)]^{-1}\sum_{n=0}^{\infty}[A_nV_1^{-n/4}D_{n+(9/2)}(2V_1^{\frac{1}{4}})]\,dV_1$$

where $(0 \leqq V_1 < \infty)$ and $m = \frac{1}{2}(n_1 - 7)$ and $D$'s are as in (2.5), (2.9) and Remark A.

(vi) *For* $p = 7$. Setting $p = 7$ in (3-A) and then making the usual substitutions for $u$'s in terms of $V$'s, as already done in case (iv) for $p = 5$, the distribution of $V_1(=u_0u_1 \cdots u_6)$ after the use of (2.1) is

$$V_1^m \exp(-\tfrac{1}{2}k_1^2)\left[2^{m+\frac{5}{2}}\pi^{\frac{3}{2}}\Gamma(m+4)\prod_{i=1}^{3}\Gamma(2m+2i)\right]^{-1} dV_1$$
$$\cdot\int_0^\infty \cdots \int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{V_1}{V_2^2} + \frac{V_2^2}{V_3^2} + \cdots + \frac{V_6^2}{V_7^2} + V_7^2\right)\right]$$
$$\times\left[1 + \frac{V_7^2}{1!}\frac{k_1^2}{2(2m+8)} + \frac{V_7^4}{2!}\frac{k_1^4}{4(2m+8)(2m+10)} + \cdots\right]\prod_{i=2}^{7} dV_i, \qquad (0 \leqq V_1 \leqq \infty).$$

To evaluate it for $V_3$ and $V_5$ we use (2.2), for $V_7$ we use both (2.2) and (2.3) and finally for $V_4$ we again use (2.3) and get the distribution of $V_1$ to be

$$(3.13)\quad \begin{aligned}&(\pi)^{\frac{1}{2}}V_1^m \exp(-\tfrac{1}{2}k_1^2)\left[2^{m+4}\Gamma(m+4)\prod_{i=1}^{3}\Gamma(2m+2i)\right]^{-1} dV_1\\ &\times\int_0^\infty\int_0^\infty\left[\sum_{n=0}^{\infty}A_n\,V_6^{(n/2)+(9/4)}V_2^{-(n/2)+(3/4)}\right]\\ &\qquad\cdot\exp\left(-\frac{1}{2}\frac{V_1}{V_2^2} - 2\left(\frac{V_2}{V_6}\right)^{\frac{1}{2}} - V_6\right)\\ &\times\left[1 + \frac{T_1''}{1!}\frac{k_1^2}{2(2m+8)} + \frac{T_2''}{2!}\frac{k_1^4}{4(2m+8)(2m+10)} + \cdots\right]dV_2\,dV_6\end{aligned}$$

where $(0 \leqq V_1 \leqq \infty)$ and $T_r'' = \sum_{n=0}^{\infty}[A_n'V_6^{r-n}]$ and $A_n'$ is as in (3.11). The distribution of $V_1(=u_0u_1 \cdots u_6)$ i.e. of the det. $|A|$ of order 7, for the noncentral linear case, is known, after evaluating (3.13) with respect of $V_2$ and $V_6$ successively and also noting that $m = \frac{1}{2}(n_1 - 8)$.

For the central case we set $k_1^2 = 0$ in (3.13) and obtain

$$(3.14)\quad \begin{aligned}&(\pi)^{\frac{1}{2}}V_1^m[2^{m+4}\Gamma(m+4)\\ &\quad\cdot\textstyle\prod_{i=1}^{3}\Gamma(2m+2i)]^{-1}dV_1\int_0^\infty\int_0^\infty[\sum_{n=0}^{\infty}A_nV_6^{(n/2)+(9/4)}V_2^{-(n/2)+(3/4)}]\\ &\quad\times\exp(-\tfrac{1}{2}V_1V_2^{-2} - 2(V_2V_6^{-1})^{\frac{1}{2}} - V_6)\,dV_2\,dV_6,\quad (0 \leqq V_1 \leqq \infty).\end{aligned}$$

After evaluating it with respect to $V_2$ and $V_6$ we obtain the distribution of

$V_1(=u_0u_1 \cdots u_6)$ i.e. of the det. $|A|$ of order 7 in the central case also where, again, $m = \frac{1}{2}(n_1 - 8)$.

(vii) *For* $p = 8$. Setting $p = 8$ in (3-A) and then making the usual substitutions for $u$'s in terms of $V$'s as done above, the distribution of $V_1(=u_0u_1 \cdots u_7)$ after the use of (2.1) is

$$2V_1^m \exp(-\tfrac{1}{2}k_1^2)\left[\pi^2 \prod_{i=1}^{4} \Gamma(2m+2i)\right]^{-1} dV_1$$
$$\times \int_0^\infty \cdots \int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{V_1}{V_2^2} + \frac{V_2^2}{V_3^2} + \cdots + \frac{V_7^2}{V_8^2} + V_8^2\right)\right]$$
$$\times \left[1 + \frac{V_8^2}{1!}\frac{k_1^2}{2(2m+9)}\right.$$
$$\left. + \frac{V_8^4}{2!}\frac{k_1^4}{4(2m+9)(2m+11)} + \cdots\right]\prod_{i=2}^{8} dV_i, \qquad (0 \leqq V_1 \leqq \infty).$$

To evaluate it for $V_2$, $V_4$, and $V_6$ we use (2.2) and for $V_8$ both (2.2) and (2.3). The distribution of $V_1$ thus is

$$\tfrac{1}{2}V_1^m \exp(-\tfrac{1}{2}k_1^2)\left[\prod_{i=1}^{4}\Gamma(2m+2i)\right]^{-1} dV_1$$
$$\times \int_0^\infty \int_0^\infty \int_0^\infty V_3 V_5 V_7 \exp\left(-\frac{V_1^{\frac{1}{2}}}{V_3} - \frac{V_3}{V_5}\right.$$
$$(3.15) \qquad \left. - \frac{V_5}{V_7} - V_7\right)\left[1 + \frac{T_1'''}{1!}\frac{k_1^2}{2(2m+9)}\right.$$
$$\left. + \frac{T_2'''}{2!}\frac{k_1^4}{4(2m+9)(2m+11)} + \cdots\right] dV_3\, dV_5\, dV_7, \quad (0 \leqq V_1 \leqq \infty),$$

where $T_r''' = \sum_{n=0}^\infty A_n' V_7^{r-n}$ and $A_n'$ is as in (3.11). The evaluation of (3.15) is easy. We first set $V_5 = t^2$ and then evaluate it with respect to $V_3$ and $V_7$ with the help of (2.3) and (2.4). This done, we finally evaluate with respect to $t$ again making use of both (2.3) and (2.4). This determines, thus, the distribution of $V_1(=u_0u_1 \cdots u_7)$ i.e. after det. $|A|$ of order 8 in the noncentral linear case where it should be remembered that $m = \frac{1}{2}(n - 9)$. For the central case, we set $k_1^2 = 0$ and $V_5 = t^2$ in (3.15) and use (2.3) to evaluate it with respect to $V_3$ and $V_7$. This gives

$$(3.16) \qquad \begin{aligned} &\pi V_1^{m+(3/8)}[\textstyle\prod_{i=1}^4 \Gamma(2m + 2i)]^{-1}\, dV_1 \int_0^\infty [\sum_{n=0}^\infty A_n V_1^{-n/4} t^{n+(11/2)}] \\ &\times [\textstyle\sum_{n=0}^\infty A_n t^{-n+\frac{3}{2}}] \exp(-2t - 2V_1^{\frac{1}{4}}t^{-1})\, dt, \quad (0 \leqq V_1 \leqq \infty), \end{aligned}$$

where $A_n$ is as in (3.6). To evaluate (3.16) with respect to $t$ we use both (2.3) and (2.4). Thus the distribution of $V_1(=u_0u_1 \cdots u_7)$ i.e. of the det. $|A|$ in the central case is known where, again, $m = \frac{1}{2}(n_1 - 9)$.

(viii) *For* $p = 9$. Setting $p = 9$ in (3-A) and then making the usual substitutions for $u$'s in terms of $V$'s the distribution of $V_1(=u_0u_1 \cdots u_8)$ after the use of (2.1) is

$$V_1^m \exp\left(-\tfrac{1}{2}k_1^2\right)\left[2^{m+3}\pi^2\Gamma(m+5)\prod_{i=1}^{4}\Gamma(2m+2i)\right]^{-1} dV_1$$

$$\cdot \int_0^\infty \cdots \int_0^\infty \exp\left[-\frac{1}{2}\left(\frac{V_1}{V_2^2}+\frac{V_2^2}{V_3^2}+\cdots+\frac{V_8^2}{V_9^2}+V_9^2\right)\right]\left[1+\frac{V_9^2}{1!}\frac{k_1^2}{2(2m+10)}\right.$$

$$\left.+\frac{V_9^4}{2!}\frac{k_1^4}{4(2m+10)(2m+12)}+\cdots\right]\prod_{i=2}^{9} dV_i, \qquad (0 \leqq V_1 \leqq \infty).$$

Following the above procedure with the use of (2.2) and (2.3), we get the distribution of $V_1$ to be

$$V_1^m \exp\left(-\tfrac{1}{2}k_1^2\right)\left[2^{m+5}\Gamma(m+5)\prod_{i=1}^{4}\Gamma(2m+2i)\right]^{-1} dV_1$$

$$\times \int_0^\infty \cdots \int_0^\infty \exp\left(-\frac{1}{2}\frac{V_1}{V_2^2}-\frac{V_2}{V_4}\right.$$

$$\text{(3.17)}\qquad \left.-\frac{V_4}{V_6}-\frac{V_6}{V_8}-V_8\right)\left[1+\frac{T_1^{(iv)}}{1!}\frac{k_1^2}{2(2m+10)}\right.$$

$$\left.+\frac{T_2^{(iv)}}{2!}\frac{k_1^4}{4(2m+10)(2m+12)}+\cdots\right] dV_2\, dV_4\, dV_6\, dV_8,$$

$$(0 \leqq V_1 \leqq \infty),$$

where $T_r^{(iv)} = \sum_{n=0}^{\infty} A_n' V_8^{r-n}$ and $A_n'$ is as in (3.11).

The evaluation of (3.23) with respect to $V_4$, $V_6$, and $V_8$ can be completed by following the same steps as explained in evaluating (3.15). Finally, we set $V_2 = (\frac{1}{2}V_1)^{\frac{1}{2}}x^{-2}$ and use (2.7), (2.11) and Remark A to evaluate the remaining integral with respect to $x(0 \leqq x \leqq \infty)$. This gives, then, the distribution of $V_1(=u_0u_1\cdots u_8)$ i.e. of the det.$|A|$ of order 9 for the noncentral linear case where it should be noted that $m = \frac{1}{2}(n_1 - 10)$.

For the central case, we set $k_1^2 = 0$ and $V_6 = t^2$ in (3.17) and use (2.3) to evaluate it with respect to $V_4$ and $V_8$. This gives:

$$\text{(3.18)}\qquad \begin{aligned}&\pi V_1^m[2^{m+4}\Gamma(m+5)\textstyle\prod_{i=1}^{4}\Gamma(2m+2i)]^{-1}\, dV_1 \int_0^\infty \int_0^\infty [\sum_{n=0}^{\infty} A_n t^{-n+\frac{3}{2}}]\\ &\times [\textstyle\sum_{n=0}^{\infty} A_n V_2^{-(n/2)+(3/4)} t^{n+(11/2)}] \exp\left(-\frac{1}{2}V_1V_2^{-2} - 2t - 2V_2^{\frac{1}{2}}t^{-1}\right) dt\, dV_2,\end{aligned}$$

where $(0 \leqq V_1 \leqq \infty)$. After evaluating it further with respect to $t$ by the use of both (2.3) and (2.4), we set $V_2 = (\frac{1}{2}V_1)^{\frac{1}{2}}x^{-2}$ and use (2.7), (2.11) and Remark A to evaluate the last integral with respect to $x(0 \leqq x \leqq \infty)$. This gives, thus the distribution of $V_1(=u_0u_1\cdots u_8)$ i.e. of the det. $|A|$ of order 9 for the central case also where it is to be kept in mind that $m = \frac{1}{2}(n_1 - 10)$.

(ix) *For* $p = 10$. Setting $p = 10$ in (3-A) and making the usual substitutions for $u$'s in terms of $V$'s, the distribution of $V_1(=u_0u_1u_2\cdots u_9)$ after the use of (2.1) is

$$2^{\frac{3}{2}}V_1^m \exp\left(-\tfrac{1}{2}k_1^2\right)\left[(\pi^{\frac{1}{2}})^5\prod_{i=1}^{5}\Gamma(2m+2i)\right]^{-1} dV_1 \cdot \int_0^\infty \cdots \int_0^\infty$$

$$\exp\left[-\frac{1}{2}\left(\frac{V_1}{V_2^2}+\frac{V_2^2}{V_3^2}+\cdots+\frac{V_9^2}{V_{10}^2}+V_{10}^2\right)\right]\left[1+\frac{V_{10}^2}{1!}\frac{k_1^2}{2(2m+11)}\right.$$
$$\left.+\frac{V_{10}^4}{2!}\frac{k_1^4}{4(2m+11)(2m+13)}+\cdots\right]\prod_{i=2}^{10}dV_i,\qquad (0\leqq V_1\leqq\infty).$$

Evaluating it first for $V_2$, $V_4$, $V_6$, $V_8$ and $V_{10}$ with the use of (2.2) and (2.3) and then after setting $V_5=t^2$ and evaluating further for $V_3$ and $V_5$ with the help of (2.3), we obtain the distribution of $V_1$ to be

$$\pi V_1^{m+(3/8)}\exp\left(-\tfrac{1}{2}k_1^2\right)\left[\prod_{i=1}^{5}\Gamma(2m+2i)\right]^{-1}dV_1$$
$$\cdot\int_0^\infty\int_0^\infty\left[\sum_{n=0}^{\infty}A_n\,t^{n+(11/2)}V_1^{-n/4}\right]$$
$$\times\left[\sum_{n=0}^{\infty}A_n\,t^{-n+(3/2)}V_9^{(n/2)+(9/2)}\right]\exp\left(-2V_1^{\frac{1}{4}}t^{-1}\right.\tag{3.19}$$
$$\left.-2t(V_9)^{\frac{1}{2}}-V_9\right)\left[1+\frac{T_1^{(v)}}{1!}\frac{k_1^2}{2(2m+11)}\right.$$
$$\left.+\frac{T_2^{(v)}}{2!}\frac{k_1^4}{4(2m+11)(2m+13)}+\cdots\right]dt\,dV_9,\qquad(0\leqq V_1\leqq\infty),$$

where $T_r^{(v)}=\sum_{n=0}^{\infty}A_n'V_9^{r-n}$ and $A_n$, $A_n'$ are as defined in (3.6) and (3.11) respectively. After evaluating (3.19) further with respect to $t$ by the use of both (2.3) and (2.4), we use (2.6), (2.10) and Remark A to evaluate the last integral with respect to $V_9$. Finally, we get the distribution of $V_1(=u_0u_1\cdots u_9)$ i.e. of the det. $|A|$ of order 10 for the noncentral linear case and where, again $m=\frac{1}{2}(n_1-11)$.

For the central case, we set $k_1^2=0$ in (3.19) and get

$$\pi V_1^{m+(3/8)}\left[\prod_{i=1}^{5}\Gamma(2m+2i)\right]^{-1}dV_1\int_0^\infty\int_0^\infty\left[\sum_{n=0}^{\infty}A_nt^{n+(11/2)}V_1^{-n/4}\right]$$
$$\times\left[\sum_{n=0}^{\infty}A_nt^{-n+\frac{3}{2}}V_9^{(n/2)+(9/4)}\right]\exp\left(-2V_1^{\frac{1}{4}}t^{-1}-2t(V_9)^{\frac{1}{2}}-V_9\right)dt\,dV_9,\tag{3.20}$$

where $(0\leqq V_1\leqq\infty)$ and $A_n$ as in (3.6). The evaluation of (3.20) with respect to $t$ and $V_9$ can be completed as explained above for the noncentral linear case. After this is done, the distribution of $V_1(=u_0u_1\cdots u_9)$ i.e. of the det. $|A|$ of order 10 is known for the central case too where $m=\frac{1}{2}(n_1-11)$.

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