DECOMPOSITION OF SYMMETRIC MATRICES AND DISTRIBUTIONS OF QUADRATIC FORMS[1]

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1. Summary and introduction. This paper presents some theorems for real matrices which should be of use and interest to anyone working with quadratic forms in normally distributed variables because of the following equivalences:

(i) stochastic independence of forms and orthogonality of their matrices (Craig [3] and Carpenter [1]),

(ii) χ^2 distribution of a form and idempotency of its matrix (Carpenter [1]), and

(iii) the distribution of a form as the unique difference of two stochastically independent χ^2 distributions and the tripotency of its matrix (i.e., $A = A^3$). The last equivalence is a consequence of the first two and the fact that every symmetric matrix is the unique difference of two non-negative, orthogonal symmetric matrices. The above equivalences apply specifically to forms in independently distributed variables with common variance, but this is inconsequential since if the random vector X has covariance matrix V, then $Y = V^{-\frac{1}{2}}X$ has covariance matrix I and $X'AX = Y'V^{\frac{1}{2}}AV^{\frac{1}{2}}Y$. Hence the matrix theorems of this paper can be easily adjusted to fit the more general case.

The matrix theorems of this paper emphasize the use of the trace of a matrix, a concept which has not been fully exploited in such results as these. As an introductory remark, we mention that throughout this manuscript all matrices are assumed to be symmetric except for those involved in Lemma 3; and our application of that lemma will be to symmetric matrices only.

2. Orthogonal matrices. The primary result of this section (Theorem 1) will generalize Cochran's Theorem. We begin with a lemma which will have application in the following section. Its proof follows easily from the fact that if A and B are non-negative, then $AB = 0$ if, and only if, tr $AB = 0$.

Lemma 1. *Let* $A = \sum_1^k A_i$ *and* $A_i \geqq 0, i = 1, \cdots, k$. *Then* $A_iA_j = 0, i \neq j, i, j = 1, \cdots, k$, *if, and only if,* $\operatorname{tr} A^2 \leqq \sum_1^k \operatorname{tr} A_i^2$.

Lemma 2 will be used to prove Theorem 1.

Lemma 2. *Let* $A = \sum_1^k A_i$. *Then* $A_iA_j = 0, i \neq j, i, j = 1, \cdots, k$, *if, and only if,* rank $A = \sum_1^k$ rank A_i *and* $A_iA_j = -A_jA_i, i \neq j, i, j = 1, \cdots, k$.

Proof. $A_iA_j = -A_jA_i, i \neq j, i, j = 1, \cdots, k$, implies that $A^2 = \sum_1^k A_i^2$ and that $A_i^2A_j^2 = A_j^2A_i^2, i \neq j, i, j = 1, \cdots, k$. Hence there is an orthogonal matrix L that simultaneously diagonalizes each A_i^2. Thus rank $A^2 = \sum_1^k$ rank A_i^2 implies that $A_i^2A_j^2 = 0$ for $i \neq j, i, j = 1, \cdots, k$. But repeated application of

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$A_iA_j = -A_jA_i$ yields $(A_iA_j)(A_iA_j)' = A_i^2A_j^2 = 0$ so that $A_iA_j = 0, i \neq j$, $i, j = 1, \cdots, k$. The converse is immediate.

THEOREM 1. *Let* $A = \sum_1^k A_i$. *Then* $A_iA_j = 0, i \neq j, i, j = 1, \cdots, k$, *if, and only if,* rank $A = \sum_1^k$ rank A_i *and* $AA_i = A_iA$, $i = 1, \cdots, k$.

PROOF. The proof is by induction: if $k = 2$, $AA_i = A_iA$ for $i = 1, 2$ implies that $A_1A_2 = A_2A_1$; hence there is an orthogonal matrix which simultaneously diagonalizes A_1 and A_2. Thus rank A = rank A_1 + rank A_2 requires that $A_1A_2 = 0$.

For $k \geqq 3$, let $A_0 = A_1 + A_2$. Then rank $A = \sum_1^k$ rank A_i and rank $A_0 \leqq$ rank A_1 + rank A_2 imply that rank A = rank $A_0 + \sum_3^k$ rank A_i. Moreover, $A_qA = AA_q$, $q = 0, 3, 4, \cdots, k$. Consequently, the induction hypothesis renders

$$A_iA_j = 0, \qquad i \neq j, i, j = 3, \cdots, k \tag{1}$$

and

$$(A_1 + A_2)A_i = A_i(A_1 + A_2) = 0, \qquad i = 3, \cdots, k. \tag{2}$$

Similarly, applying the induction hypothesis to each of $A = A_1 + (A_2 + A_3) + A_4 + \cdots + A_k$ and $A = (A_1 + A_3) + A_2 + A_4 + \cdots + A_k$, we obtain

$$A_1A_i = 0, \qquad i = 4, \cdots, k, \tag{3}$$

$$A_i(A_2 + A_3) = (A_2 + A_3)A_i = 0, \qquad i = 1, 4, \cdots, k, \tag{4}$$

$$A_2A_i = 0, \qquad i = 4, \cdots, k \tag{5}$$

and

$$A_i(A_1 + A_3) = (A_1 + A_3)A_i = 0, \qquad \text{i} = 2, 4, \cdots, k. \tag{6}$$

Summarizing, from (1)–(6) we obtain $A_iA_j = 0$ for $i \neq j$, $i = 1, \cdots, k$, $j = 4, \cdots, k$, and $A_iA_j = -A_jA_i$ for $i \neq j$, $i, j = 1, 2, 3$. Thus the desired result follows from Lemma 2.

Theorem 1 is a generalization of Cochran's Theorem (see Cochran [2] and Madow [6]): let $A = I$ to obtain the latter. Moreover, the following algebraic lemma shows that one may assume $A = I$ in the proof of Graybill and Marsaglia's generalization of Cochran's Theorem ([4], Theorem 1(b)); hence that theorem follows from Cochran's Theorem and the upcoming lemma. This lemma will have further use in the sequel.

LEMMA 3. *Let* $A = \sum_1^k A_i$ *where* A *and* A_i, $i = 1, \cdots, k$, *are* $m \times n$ *matrices* (*hence not necessarily symmetric*), *and let* rank $A = \sum_1^k$ rank A_i. *Then for* $1 \leqq q \leqq m$, *if the* qth *row vector of* A *is the zero vector, so is the* qth *row vector of* A_i *for each* $i = 1, \cdots, k$.

PROOF. By induction we may assume $k = 2$; hence we show that if a_1 and a_2 denote the qth row vectors of A_1 and A_2, respectively, then a_1 and a_2 are zero vectors if $a_1 + a_2$ is the zero vector.

Let r = rank A_1, p = rank A_2. Let $x_1, \cdots, x_r$ and $y_1, \cdots, y_p$ be bases for the row spaces of A_1 and A_2, respectively. Then $a_1 = \sum_1^r s_jx_j$ and $a_2 =$

$\sum_1^p t_j y_j$ for certain scalars s_j, $j = 1, \cdots, r$, and t_j, $j = 1, \cdots, p$. But $a_1 = -a_2$ so if a_1 is not the zero vector there is a j_0, $1 \leqq j_0 \leqq r$, such that x_{j_0} is a linear combination of the $r + p - 1$ vectors $\{x_j; j = 1, \cdots, r, j \neq j_0\} \cup \{y_j; j = 1, \cdots, p\}$. It follows that every row vector of A is also a linear combination of these $r + p - 1$ vectors. Hence rank $A \leqq r + p - 1 <$ rank A_1 + rank A_2 which is a contradiction.

3. Idempotent and orthogonal matrices. Let $A = \sum_1^k A_i$; we intend to exploit the concept of the trace of a matrix to give necessary and sufficient conditions for the idempotency and pairwise orthogonality of the matrices $A_1, \cdots, A_k$. In this section and the next, we will repeatedly use without mention the following well-known facts and their consequences:

(i) $A = A^2$ if, and only if, rank $A = \text{tr } A = \text{tr } A^2$, and

(ii) $(\text{tr } A)^2 \leqq (\text{rank } A)(\text{tr } A^2)$.

These facts are direct consequences of applying the Cauchy-Schwarz Inequality (and its statement of equality) to the non-zero proper values of A; and (i), in particular, suggests the possible usefulness of the trace for the purpose of this section as well as the next.

THEOREM 2. *Let* $A = \sum_1^k A_i$; *the following are equivalent.*

(i) $A_i = A_i^2$, $i = 1, \cdots, k$, *and* $A_iA_j = 0$, $i \neq j$, $i, j = 1, \cdots, k$.

(ii) $A = A^2$, $A_i \geqq 0$, $i = 1, \cdots, k$, *and* $\text{tr } A \leqq \sum_1^k \text{tr } A_i^2$.

(iii) $A = A^2$, $\text{tr } A_iA_j \geqq 0$, $i \neq j$, $i, j = 1, \cdots, k$, *and for each* $i = 1, \cdots, k-1$, *at least one of the following conditions holds*:

(a) rank $A_i \leqq \text{tr } A_i$,

(b) rank $A_i \leqq \text{tr } A_i^2$, *or*

(c) $A_i = A_i^2$.

(iv) $A = A^2$, rank $A_i \leqq \text{tr } A_i$, $i = 1, \cdots, k-1$, *and* $\text{tr } (A - A_k)A_k \geqq 0$ *or* $\text{tr } A_iA_k \geqq 0$, $i = 1, \cdots, k-1$.

(v) $A = A^2$, rank $A_i \leqq \text{tr } A_i$, $i = 1, \cdots, k-1$, *and* $\text{tr } (\sum_{i=1}^k \sum_{j=1, i\neq j}^k A_iA_j) \geqq 0$ *or* $\text{tr } A \geqq \sum_1^k \text{tr } A_i^2$.

(vi) $A = A^2$, $A_i = A_i^2$, $i = 1, \cdots, k-1$, *and* $\text{tr } A_k \geqq \text{tr } A_k^2$.

(vii) $A = A^2$, $\text{tr } A_i \leqq \text{tr } A_i^2$, $i = 1, \cdots, k-1$, *and* $A_i \geqq 0$, $i = 1, \cdots, k$.

(viii) $A = A^2$, rank $A_i \leqq \text{tr } A_i^2$, $i = 1, \cdots, k-1$, *and* $A_i \geqq 0$, $i = 1, \cdots, k$.

(ix) $A = A^2$, rank $A_i \leqq \text{tr } A_i$, i $= 1, \cdots, k-1$, *and* $A_k \geqq 0$.

(x) $A = A^2$, $A_i \geqq 0$, $i = 1, \cdots, k$, *and for each* $i = 1, \cdots, k-1$, *at least one of the following conditions holds*:

(a) rank $A_i \leqq \text{tr } A_i$,

(b) rank $A_i \leqq \text{tr } A_i^2$,

(c) $\text{tr } A_i \leqq \text{tr } A_i^2$, *or*

(d) $A_i = A_i^2$.

PROOF. Obviously (i) implies all others, (viii) implies (vii), and (x) implies (vii). Moreover, (vi) implies (v) since $\text{tr } A = \sum_1^k \text{tr } A_i = \sum_1^{k-1} \text{tr } A_i^2 + \text{tr } A_k \geqq \sum_1^k \text{tr } A_i^2$, and (ii) implies (i) by Lemma 1. Hence it suffices to show that each of (iii), (iv), (v), (vii), and (ix) implies (*i*).

(iii) implies (i): It suffices to prove the result assuming rank $A_i \leqq \text{tr } A_i^2$,

$i = 1, \cdots, k-1$. The proof is inductive: For $k = 2$, by diagonalizing with a suitable orthogonal matrix we may assume that $A = A_1 + A_2$ has the form

$$\begin{pmatrix} I_u & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \bar\lambda_1 & D' \\ D & \bar\lambda_2 \end{pmatrix} + \begin{pmatrix} I_u - \bar\lambda_1 & -D' \\ -D & -\bar\lambda_2 \end{pmatrix},$$

where $0 \leqq u \leqq n$ and $\bar\lambda_1, \bar\lambda_2$ are diagonal matrices. Let $r = \operatorname{rank} A_1$. Then $\operatorname{tr} A_1^2 \geqq r$ requires that

(1) $\operatorname{tr} \bar\lambda_1^2 + \operatorname{tr} \bar\lambda_2^2 + 2 \operatorname{tr} DD' \geqq r$

and $\operatorname{tr} A_1A_2 \geqq 0$ implies that

(2) $\operatorname{tr} \bar\lambda_1 - \operatorname{tr} \bar\lambda_1^2 - \operatorname{tr} \bar\lambda_2^2 - 2 \operatorname{tr} DD' \geqq 0$.

The addition of (1) and (2) yields $\operatorname{tr} \bar\lambda_1 \geqq r$ so that $\operatorname{tr} \bar\lambda_1 \geqq \operatorname{rank} \bar\lambda_1$. But, using (2) again, we obtain $\operatorname{tr} \bar\lambda_1 \geqq \operatorname{tr} \bar\lambda_1^2 + \operatorname{tr} \bar\lambda_2^2 + 2 \operatorname{tr} DD' \geqq \operatorname{tr} \bar\lambda_1^2 + \operatorname{tr} \bar\lambda_2^2 \geqq \operatorname{tr} \bar\lambda_1^2$ so that $\operatorname{rank} \bar\lambda_1 = \operatorname{tr} \bar\lambda_1 = \operatorname{tr} \bar\lambda_1^2$, $D = 0$, and $\bar\lambda_2 = 0$. The desired result follows.

For $k \geqq 3$, let $A = A_1 + A_0$, where $A_0 = \sum_2^k A_i$. Then $A = A^2$, $\operatorname{tr} A_1A_0 = \sum_2^k \operatorname{tr} A_1A_j \geqq 0$, and $\operatorname{rank} A_1 \leqq \operatorname{tr} A_1^2$ imply that $A_0 = A_0^2$ and $A_1 = A_1^2$. Hence $A_i = A_i^2$, $i = 2, \cdots, k$, by the induction hypothesis. It follows that $A_iA_j = 0$, $i \neq j$, $i, j = 1, \cdots, k$, and the proof is complete.

(iv) implies (i): Let $A = A_0 + A_k$, where $A_0 = \sum_{i=1}^{k-1} A_i$. We have $\operatorname{rank} A_0 \leqq \sum_1^{k-1} \operatorname{rank} A_i \leqq \sum_{i=1}^{k-1} \operatorname{tr} A_i = \operatorname{tr} A_0$ and $\operatorname{tr} A_0A_k \geqq 0$. Hence $A_0 = A_0^2$ and $A_k = A_k^2$ by the fact that (iii) implies (i). Hence $\operatorname{rank} A_0 = \operatorname{tr} A_0 = \sum_1^{k-1} \operatorname{tr} A_i \geqq \sum_1^{k-1} \operatorname{rank} A_i$ implies that $A_i = A_i^2$, $i = 1, \cdots, k-1$, by Graybill and Marsaglia's generalization of Cochran's Theorem ([4], Theorem 1(b)) that was mentioned earlier. Hence the desired result follows.

(v) implies (i): The proof is by induction: For $k = 2$, (v) is contained in (iii). For $k \geqq 3$, since

$$\sum_{i=1}^k \sum_{j=1, i\neq j}^k \operatorname{tr} A_iA_j = \sum_{j=1}^{k-1} \operatorname{tr} A_kA_j + \sum_{i=1}^{k-1} \sum_{j=1, i\neq j}^k \operatorname{tr} A_iA_j \geqq 0,$$

we consider two cases:

Case 1: If $\sum_{j=1}^{k-1} \operatorname{tr} A_kA_j \geqq 0$, the result follows from the fact that (iv) implies (i).

Case 2: If $\sum_{i=1}^{k-1} \sum_{j=1, j\neq i}^k \operatorname{tr} A_iA_j \geqq 0$, there is an i_0, $1 \leqq i_0 \leqq k-1$, such that $\sum_{j=1, j\neq i_0}^k \operatorname{tr} A_{i_0}A_j \geqq 0$. Hence if we let $A = A_{i_0} + A_0$, where $A_0 = \sum_{j=1, j\neq i_0}^k A_j$, we have $\operatorname{rank} A_{i_0} \leqq \operatorname{tr} A_{i_0}$ and $\operatorname{tr} A_{i_0}A_0 \geqq 0$; since (iii) implies (i) it follows that $A_{i_0} = A_{i_0}^2$, $A_0 = A_0^2$, and $A_{i_0}A_0 = 0$. Consequently, $\sum_{j=1, j\neq i_0}^k \operatorname{tr} A_{i_0}A_j = 0$, which implies that

$$\sum_{\substack{i=1 \\ i\neq j, i\neq i_0, j\neq i_0}}^k \sum_{j=1}^k \operatorname{tr} A_iA_j = \sum_{i=1}^k \sum_{\substack{j=1 \\ j\neq i}}^k \operatorname{tr} A_iA_j - 2 \sum_{j=1, j\neq i_0}^k \operatorname{tr} A_{i_0}A_j \geqq 0.$$

Thus $A_i = A_i^2$, $i \neq i_0$, $i = 1, \cdots, k$, by the induction assumption.

(vii) implies (i): Since $A_i \geqq 0$, $i = 1, \cdots, k$, by diagonalizing with an orthogonal matrix we can assume $A = I$. Hence, $A_j = \sum_{i=1}^k A_iA_j$, $j = 1, \cdots, k-1$ so that $\operatorname{tr} A_j = \sum_{i=1, i\neq j}^k \operatorname{tr} A_iA_j + \operatorname{tr} A_j^2 \geqq \sum_{i=1, i\neq j}^k \operatorname{tr} A_iA_j + \operatorname{tr} A_j$,

$j = 1, \cdots, k - 1$, implies that $\sum_{i=1, i\neq j}^{k} \operatorname{tr} A_iA_j \leqq 0, j = 1, \cdots, k - 1$. It follows that $A_iA_j = 0, i \neq j, i, j = 1, \cdots, k$.

(ix) implies (i): Consider first the case $k = 2$. We can assume $A = A_1 + A_2$ has the form

$$\begin{pmatrix} I_u & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \bar{\lambda}_1 & D \\ D' & E \end{pmatrix} + \begin{pmatrix} I_u - \bar{\lambda}_1 & -D \\ -D' & -E \end{pmatrix},$$

where $\bar{\lambda}_1$ is a diagonal $u \times u$ matrix $(0 \leqq u \leqq n)$ with diagonal elements $\lambda_1, \cdots, \lambda_u$. Now $A_2 \geqq 0$ implies that $\lambda_i \leqq 1, i = 1, \cdots, u$. We shall renumber the $\lambda_1, \cdots, \lambda_u$ so that $\lambda_1, \cdots, \lambda_r$ are the non-zero ones. Then $\operatorname{tr} \bar{\lambda}_1 = \sum_1^r \lambda_i \leqq r = \operatorname{rank} \bar{\lambda}_1$. On the other hand, $\operatorname{rank} \bar{\lambda}_1 \leqq \operatorname{rank} A_1 \leqq \operatorname{tr} A_1 \leqq \operatorname{tr} \bar{\lambda}_1$, the last inequality holding since $A_2 \geqq 0$. Hence, $\operatorname{rank} \bar{\lambda}_1 = \operatorname{tr} A_1 = \operatorname{tr} \bar{\lambda}_1 = \sum_1^r \lambda_i$. Thus $\lambda_i = 1, i = 1, \cdots, r, D = 0$, and $E = 0$ so that $A_1 = A_1^2, A_2 = A_2^2$, and $A_1A_2 = 0$.

In the general case $(k \geqq 2)$, let $A = A_0 + A_k$ where $A_0 = \sum_1^{k-1} A_i$. Now $\operatorname{rank} A_0 \leqq \sum_1^{k-1} \operatorname{rank} A_i \leqq \sum_1^{k-1} \operatorname{tr} A_i = \operatorname{tr} A_0$ and $A_k \geqq 0$. It follows from the case for $k = 2$ that $A_0 = A_0^2$ and $A_k = A_k^2$. Hence, since $\operatorname{rank} A_i \leqq \operatorname{tr} A_i$, $i = 1, \cdots, k - 1$, it follows from Graybill and Marsaglia's generalization of Cochran's Theorem, as in the proof of (iv) implies (i), that $A_i = A_i^2$, $i = 1, \cdots, k - 1$. The proof is complete.

It should be noted that the Hogg-Craig Theorem [5] is an immediate corollary of $p \Rightarrow$ (i) of the preceding theorem for every $p =$ (iii), $\cdots$, (x), except $p =$ (vi). Moreover, one can easily construct examples which show that the hypotheses in conditions (ii)–(x) are essentially minimal. For example, it is essential that every A_i be non-negative in the hypotheses of conditions (vii) and (viii).

4. Tripotent and orthogonal matrices. Let $A = \sum_1^k A_i$; we will play the same theme as in the preceding section with the exception that here we are interested in the tripotency and orthogonality of the matrices $A_1, \cdots, A_k$. We list some well-known and easily-verified results which will be used throughout this section:

(i) if $A_iA_j = 0, i \neq j, i, j = 1, \cdots, k$, then $A = A^3$ if, and only if, $A_i = A_i^3$, $i = 1, \cdots, k$;

(ii) there are unique non-negative symmetric matrices B and C such that $BC = 0$ and $A = B - C$; indeed, if $A = A^3$, $B = (A^2 + A)/2$ and $C = (A^2 - A)/2$ are idempotent;

(iii) the following are equivalent:

(a) $A = A^3$,

(b) every proper value of A has value -1, 0, or 1,

(c) $A^i = A^{i+2}$ for every $i = 1, 2, \cdots$,

(d) $\operatorname{rank} A = \operatorname{tr} A^2 = \operatorname{tr} A^4$.

We need two preliminary lemmas.

Lemma 4. *Let* $A = A_1 + A_2$. *If* $A = A^3$, $A_1 \geqq 0$, $\operatorname{tr} A_1A_2 \geqq 0$, *and either* rank $A_1 \leqq \operatorname{tr} A_1$ *or* rank $A_1 \leqq \operatorname{tr} A_1^2$, *then* $A_1 = A_1^2$, $A_2 = A_2^3$, *and* $A_1A_2 = 0$.

PROOF. It suffices to prove the result assuming rank $A_1 \leqq \operatorname{tr} A_1^2$. By an orthogonal diagonalization of A, we may assume $A = A_1 + A_2$ has the form

$$\begin{pmatrix} I_1 & 0 & 0 \\ 0 & -I_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix} + \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

Then $A_1 A = A_1^2 + A_1 A_2$ and $\operatorname{tr} A_1 A_2 \geqq 0$ imply that $\operatorname{tr} A_1^2 \leqq \operatorname{tr} A_1 A = \operatorname{tr} B_{11} - \operatorname{tr} B_{22} \leqq \operatorname{tr} B_{11} \leqq \operatorname{tr} A_1$. Together with rank $A_1 \leqq \operatorname{tr} A_1^2$, this implies that rank $A_1 = \operatorname{tr} A_1 = \operatorname{tr} A_1^2$, hence $A_1 = A_1^2$ and $\operatorname{tr} A_1 = \operatorname{tr} B_{11}$. From the latter it follows that $\operatorname{tr} B_{22} = \operatorname{tr} B_{33} = 0$. Thus $B_{ij} = 0$ unless $i = j = 1$ so that $C_{ij} = 0$ unless $i = j = 1$ or 2; hence $B_{11} = B_{11}^2$, $C_{22} = -I_2$, and $C_{11} = I_1 - B_{11}$. The conclusion follows.

LEMMA 5. *Let* $A = A_1 + A_2$. *If* $A = A^3$, $A_1 \geqq 0$, $\operatorname{tr} A_1 \leqq$ rank A_1, *and* rank $A =$ rank $A_1 +$ rank A_2, *then* $A_1 = A_1^2$, $A_2 = A_2^3$, *and* $A_1 A_2 = 0$.

PROOF. Using Lemma 3 after diagonalizing with a suitable orthogonal matrix, we may assume that $A = A_1 + A_2$ has the form

$$\begin{pmatrix} I_u & 0 \\ 0 & -I_v \end{pmatrix} = \begin{pmatrix} \bar\lambda_1 & D_1' \\ D_1 & \bar\lambda_2 \end{pmatrix} + \begin{pmatrix} I_u - \bar\lambda_1 & -D_1' \\ -D_1 & -I_v - \bar\lambda_2 \end{pmatrix},$$

where $\bar\lambda_1$ is a diagonal matrix with non-negative diagonal elements $\lambda_1, \cdots, \lambda_u$ and $\bar\lambda_2$ is a diagonal matrix with non-negative diagonal elements $\lambda_{u+1}, \cdots, \lambda_{u+v}$.

Let p be the number of λ_i, $1 \leqq i \leqq u$, which have value 0 ($0 \leqq p \leqq u$). Since $A_1 \geqq 0$, we may now assume that $A = A_1 + A_2$ has the form

$$\begin{pmatrix} I_u & 0 \\ 0 & -I_v \end{pmatrix} = \begin{pmatrix} \bar\gamma_1 & 0 & D' \\ 0 & 0 & 0 \\ D & 0 & \bar\lambda_2 \end{pmatrix} + \begin{pmatrix} I_r - \bar\gamma_1 & 0 & -D' \\ 0 & I_p & 0 \\ -D & 0 & -I_v - \bar\lambda_2 \end{pmatrix},$$

where $r + p = u$, and $\bar\gamma_1 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_r \end{pmatrix}$ has rank r. Thus rank $A_1 \geqq r$. But $-1 - \lambda_i \leqq -1$, $i = u + 1, \cdots, u + v$; hence the last $v + p$ rows of A_2 are linearly independent. Consequently, rank $A =$ rank $A_1 +$ rank A_2 implies that rank $A_1 = r$ and rank $A_2 = v + p$. Thus the last $v + p$ row vectors of A_2 form a basis for the row space of A_2. Hence for each $q = 1, \cdots, r$, there are scalars t_{iq}, $i = r + 1, \cdots, u + v$, such that $a_q = \sum_{i=r+1}^{u+v} t_{iq} a_i$, where a_j denotes the jth row vector of A_2, $j = 1, \cdots, u + v$. But necessarily $t_{iq} = 0$, $i = r + 1, \cdots, r + p$, $q = 1, \cdots, r$; hence $a_q = \sum_{i=u+1}^{u+v} t_{iq} a_i$, $q = 1, \cdots, r$. If we let $D = (d_{ij})$, $i = u + 1, \cdots, u + v$, $j = 1, \cdots, r$, it follows that for each $q = 1, \cdots, r$, $(0, \cdots, 0, 1 - \lambda_q, 0, \cdots, 0, -d_{u+1,q}, \cdots, -d_{u+v,q}) = \sum_{i=u+1}^{u+v} t_{iq}(-d_{i,1}, \cdots, -d_{i,r}, 0, \cdots, 0, -1 - \lambda_i, 0, \cdots, 0)$. Thus $\delta_{qk}(1 - \lambda_q) = -\sum_{i=u+1}^{u+v} t_{iq}\, d_{i,k}$, $q, k = 1, \cdots, r$ (δ_{qk} denotes the Kronecker delta), and $-d_{i,q} = t_{iq}(-1 - \lambda_i)$,

$i = u + 1, \cdots, u + v$, $q = 1, \cdots, r$. Consequently, $1 - \lambda_q = \sum_{i=u+1}^{u+v} t_{iq}^2(-1 - \lambda_i)$, $q = 1, \cdots, r$. Hence from the fact that $-1 - \lambda_i \leqq -1$, $i = u + 1, \cdots, u + v$, we can conclude that for each $q = 1, \cdots, r$, $1 - \lambda_q \leqq 0$ and indeed $1 - \lambda_q = 0$ if, and only if, $t_{iq} = 0$, $i = u + 1, \cdots, u + v$. Accordingly, $\operatorname{rank} A_1 \leqq \sum_1^r \lambda_q \leqq \operatorname{tr} A_1$. But $\operatorname{tr} A_1 \leqq \operatorname{rank} A_1$ by hypothesis; hence, $\lambda_q = 1$, $q = 1, \cdots, r$. It follows that $t_{iq} = 0$, $i = u + 1, \cdots, u + v$, $q = 1, \cdots, r$. Thus $D = 0$ from which it follows that $\bar{\lambda}_2 = 0$. The proof is complete.

THEOREM 3. *Let* $A = \sum_1^k A_i$ *; the following are equivalent.*

(i) $A_i = A_i^3$, $i = 1, \cdots, k$, *and* $A_iA_j = 0$, $i \neq j$, $i, j = 1, \cdots, k$.

(ii) $A = A^3$, $A_i = A_i^3$, $i = 1, \cdots, k - 1$, *and* $\operatorname{tr} A_iA_j \geqq 0$, $i \neq j$, $i, j = 1, \cdots, k$.

(iii) $A = A^3$, $\operatorname{rank} A = \sum_1^k \operatorname{rank} A_i$, *and* $\operatorname{tr} A_i^2 \leqq \operatorname{rank} A_i$, $i = 1, \cdots, k - 1$.

(iv) $A = A^3$, $A_i = A_i^3$, $i = 1, \cdots, k - 1$, *and* $\operatorname{rank} A = \sum_1^k \operatorname{rank} A_i$.

(v) $A = A^3$, $\operatorname{rank} A = \sum_1^k \operatorname{rank} A_i$, *and* $\operatorname{tr} A_iA_j \geqq 0$, $i \neq j$, $i, j = 1, \cdots, k$.

PROOF. It is immediate that (i) implies all others and that (iv) implies (iii). Hence it remains to show that each of (ii), (iii) and (v) implies (i).

(ii) implies (i): The proof is by induction. For $k = 2$, let $A_1 = B_1 + B_2$, where $B_1 = B_1^2$, $-B_2 = B_2^2$, and $B_1B_2 = 0$. Now $\operatorname{tr}(B_1 + B_2)A_2 = \operatorname{tr} B_1A_2 + \operatorname{tr} B_2A_2 \geqq 0$ by hypothesis. Thus either $\operatorname{tr} B_1A_2 \geqq 0$ or $\operatorname{tr} B_2A_2 \geqq 0$. Suppose $\operatorname{tr} B_1A_2 \geqq 0$. (The proof for $\operatorname{tr} B_2A_2 \geqq 0$ is similar.) Then $\operatorname{tr} B_1(B_2 + A_2) = \operatorname{tr} B_1A_2 \geqq 0$; hence, $B_2 + A_2 = (B_2 + A_2)^3$ and $B_1A_2 = B_1(B_2 + A_2) = 0$ by Lemma 4. Consequently, $\operatorname{tr} B_1A_2 = 0$ so that $\operatorname{tr}(-B_2)(-A_2) = \operatorname{tr} B_2A_2 \geqq 0$. Applying Lemma 4 to $-(B_2 + A_2) = -B_2 + (-A_2)$, it follows that $A_2 = A_2^3$ and $B_2A_2 = 0$. Hence $A_1A_2 = 0$.

For $k \geqq 3$, let $A = A_j + B_j$, $j = 1, \cdots, k - 1$, where $B_j = \sum_{i=1, i\neq j}^k A_i$, $j = 1, \cdots, k - 1$. Then $\operatorname{tr} A_jB_j \geqq 0$, $j = 1, \cdots, k - 1$; hence $B_j = B_j^3$ and $A_jB_j = 0$, $j = 1, \cdots, k - 1$. Thus the induction assumption implies that $A_k = A_k^3$ and that for each $j = 1, \cdots, k - 1$, $A_iA_q = 0$, $i \neq q$, $i \neq j$, $q \neq j$, $i, q = 1, \cdots, k$. It follows that $A_iA_j = 0$, $i \neq j$, $i, j = 1, \cdots, k$.

(iii) implies (i): Again induction is used: Suppose $k = 2$. Let $A_1 = B_1 + B_2$ where $B_1 \geqq 0$, $B_2 \leqq 0$, and $B_1B_2 = 0$. Consequently, $A_1^2 = B_1^2 + B_2^2$; it follows that $\operatorname{tr} B_1^2 + \operatorname{tr} B_2^2 = \operatorname{tr} A_1^2 \leqq \operatorname{rank} A_1 = \operatorname{rank} B_1 + \operatorname{rank} B_2$. Accordingly, either $\operatorname{tr} B_1^2 \leqq \operatorname{rank} B_1$ or $\operatorname{tr} B_2^2 \leqq \operatorname{rank} B_2$. Suppose $\operatorname{tr} B_1^2 \leqq \operatorname{rank} B_1$. (The proof for $\operatorname{tr} B_2^2 \leqq \operatorname{rank} B_2$ is similar.) Since $\operatorname{rank} A = \operatorname{rank} B_1 + \operatorname{rank}(B_2 + A_2)$, it follows from Lemma 5 that $B_1 = B_1^2$, $(B_2 + A_2) = (B_2 + A_2)^3$, and $B_1(B_2 + A_2) = B_1A_2 = 0$. Thus $\operatorname{tr} B_1^2 = \operatorname{rank} B_1$ which implies that $\operatorname{tr} B_2^2 \leqq \operatorname{rank} B_2$. Moreover, $\operatorname{rank}(B_2 + A_2) = \operatorname{rank} B_2 + \operatorname{rank} A_2$. Hence if we apply Lemma 5 to $-(B_2 + A_2) = -B_2 + (-A_2)$, we obtain $-B_2 = B_2^2$, $A_2 = A_2^3$, and $B_2A_2 = 0$. Consequently, $A_1 = A_1^3$ and $A_1A_2 = 0$. The proof for $k \geqq 3$ is quite similar to that for (ii) implies (i).

(v) implies (i): We again use induction. Let $k = 2$. Then $\operatorname{rank} A_1 + \operatorname{rank} A_2 = \operatorname{rank} A = \operatorname{tr} A^2 = \operatorname{tr} A_1^2 + \operatorname{tr} A_2^2 + 2 \operatorname{tr} A_1A_2 \geqq \operatorname{tr} A_1^2 + \operatorname{tr} A_2^2$. Accordingly, either $\operatorname{tr} A_1^2 \leqq \operatorname{rank} A_1$ or $\operatorname{tr} A_2^2 \leqq \operatorname{rank} A_2$. Thus the desired result follows from (iii) implies (i). The proof for $k \geqq 3$ is as before.

It is easy to show by example that the hypotheses in (ii)–(v) are essentially minimal. In particular, the direct analogy of Cochran's Theorem is not true: the hypotheses $A = A^3$ and rank $A = \sum_1^k$ rank A_i are not sufficient to guarantee that (i) is valid. Moreover, one cannot delete either the condition $A_i = A_i^3$, $i = 1, \cdots, k - 1$, or the condition tr $A_iA_j \geqq 0$, $i \neq j$, $i, j = 1, \cdots, k$, from (ii) and still have (ii) imply (i).

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