# EFFECT OF NON-NORMALITY ON STEIN'S TWO SAMPLE TEST

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**1. Introduction.** Student's $t$ test is used to test the hypothesis about the mean of a normal population when the variance is not known; the power of this test being dependent on the unknown variance. It is shown by Dantzig (1940) that for a sample of fixed size there does not exist a test for Student's hypothesis whose power is independent of the variance. Stein (1945) gave a two sample test for a linear hypothesis whose power is independent of the unknown variance. He used it (i) to test the hypothesis about the mean of a normal population and (ii) to estimate the mean by a confidence interval of predetermined length with a given confidence coefficient. As in other tests of significance the basic assumptions in Stein's test is the normality of the parent population. This assumption of normality may not hold good in practice, and hence the validity of normal theory Stein's test for non-normal populations should be examined.

The effect of non-normality on Student's test has been investigated, among others, by Pearson and Adyanthaya (1929), Bartlett (1935), Geary (1936), Gayen (1949), Ghurye (1949) and Srivastava (1958). Pearson and Adyanthaya (1929) have shown by some experimental investigation that the effect of skewness and kurtosis of the parent population on Student's $t$ may be considerable. Bartlett (1935) confirmed Pearson's results theoretically by obtaining an approximate distribution of $t$ in non-normal samples. Assuming the parent population to be represented by the first two terms of an Edgeworth series, Geary (1936) obtained the approximate distribution of $t$. Gayen (1949) considered the effect of both skewness ($\lambda_3$) and kurtosis ($\lambda_4$) by using the first four terms of the Edgeworth series as the frequency function of the population to derive the distribution of $t$. A theoretical study on the effect of non-normality on the power of the $t$ test was first made by Ghurye (1949) by considering the first two terms of the Edgeworth series and later Srivastava (1958) extended this work by considering the effects of $\lambda_4$ and $\lambda_3^2$. In a recent paper, Bhattacharjee and Nagendra (1964) have studied the effect of non-normality on the Wald sequential test for mean. This will be of particular interest as Stein's test can be considered as a special case of sequential test.

In this paper, the effect of non-normality on Stein's two sample scheme is investigated by deriving the distribution of Stein's $t$ for non-normal populations represented by the first four terms of an Edgeworth series. The power function of Stein's test and the confidence level of the fixed length confidence interval are also obtained.

**2. Stein's two sample scheme.** Stein's (1945) two sample procedure for (i) testing the mean of a normal population with unknown variance and (ii) for

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estimating the mean by a fixed length confidence interval may be stated as follows:

Let $x_1, x_2, \cdots, x_{n_0}$ be a random sample of size $n_0$ from a normal population with mean $\mu$ and variance $\sigma^2$, and the estimate of the variance be

$$s_0^2 = (n_0 - 1)^{-1} \sum_1^{n_0} (x_i - \bar{x}_0)^2,$$

where

$$\bar{x}_0 = (n_0)^{-1} \sum_1^{n_0} x_i .$$

Let $x_{n_0+1}, x_{n_0+2}, \cdots, x_n$ be a second sample of size $n - n_0$ ($= n_1$ say) from the same population,
where

$$n = \max \{[s_0^2/z] + 1, n_0\}.$$

$z$ being a preassigned positive constant and $[q]$ denoting the largest integer less than $q$. Let

$$\begin{aligned} \bar{x}_1 &= (n_1)^{-1} \sum_1^{n_1} x_{n_0+i}, \\ \bar{x} &= (n_0\bar{x}_0 + n_1\bar{x}_1)/n, \qquad (1) \\ t &= (\bar{x} - \mu)/(s_0/n^{\frac{1}{2}}) \end{aligned}$$

The statistic $t$ follows Student's $t$ distribution with $n_0 - 1$ ($= \nu$ say) degrees of freedom.

(i) An unbiased test for the hypothesis $H_0(\mu = \mu_0)$ can be obtained by rejecting $H_0$ if $|(\bar{x} - \mu_0)/(s_0/n^{\frac{1}{2}})| > t(\alpha/2, \nu)$, where $t(\alpha, \nu)$ is the upper $100\alpha$ percentage point of Student's $t$ distribution. The power function of this test is given by

$$\phi(\mu) = 1 - \beta(\mu),$$

where $\beta(\mu) = \text{Prob}\{-t(\alpha/2, \nu) + (\mu_0 - \mu)/z^{\frac{1}{2}} < t_\nu < t(\alpha/2, \nu) + (\mu_0 - \mu)/z^{\frac{1}{2}}\}$; $t_\nu$ being a Student's $t$ variate with $\nu$ degrees of freedom. Similarly, $H_0$ can be tested against one sided alternatives $\mu > \mu_0$.

(ii) A confidence interval for $\mu$ of predetermined length $d$ and confidence coefficient $1 - \alpha$ can be obtained by selecting $z = d^2/\{4t^2(\alpha/2, \nu)\}$ such that

$$1 - \alpha = \text{Prob}\{\bar{x} - d/2 < \mu < \bar{x} + d/2\}.$$

**3. Distribution of Stein's $t$ in non-normal samples.** If the parent population is not normal the statistic $t$ defined in (1) will not follow Student's $t$ distribution. We propose here to derive the distribution of the statistic $t$ when the underlying population is represented by the first four terms of an Edgeworth series, as

$$\begin{aligned} (2) \quad f(x) = \sigma^{-1}\{&\phi((x - \mu)/\sigma) - (\lambda_3/3!)\phi^{(3)}((x - \mu)/\sigma) \\ &+ (\lambda_4/4!)\phi^{(4)}((x - \mu)/\sigma) + (\lambda_3^2/72)\phi^{(6)}((x - \mu)/\sigma)\}, \end{aligned}$$

where $\lambda_3$ ($= \beta_1^{\frac{1}{2}}$) and $\lambda_4$ ($= \beta_2 - 3$) are the measures of skewness and kurtosis respectively, $\phi(x) = (2\pi)^{-\frac{1}{2}}e^{-x^2/2}$ and $\phi^{(r)}(x)$ is the $r$th derivative of $\phi(x)$.

In Stein's procedure, the total sample size $n$ is either equal to $n_0$ or greater than $n_0$ according as $s_0^2 <$ or $\geqq n_0 z$ ($= c$ *say*). To obtain the joint distribution of $\bar{x}$ and $s_0^2$, we divide the sample space into two pieces say $B$ and $\bar{B}$ according as $n = n_0$ or $n > n_0$.

In $B$; $n = n_0$, $\bar{x} = \bar{x}_0$ and the distribution of $\bar{x}$ and $s_0^2$ is the same as that obtained by Gayen (1949) for $\bar{x}_0$ and $s_0^2$ (*say*, $g_0(\bar{x}_0, s_0^2)$).

In $\bar{B}$; $n = n_0 + n_1$, $\bar{x} = (n_0\bar{x}_0 + n_1\bar{x}_1)/n$ and the joint distribution of $\bar{x}$ and $s_0^2$ is obtained by considering the distributions of $\bar{x}_0$, $\bar{x}_1$ and $s_0^2$ as follows:

Denoting density functions of random variables $u_1, u_2, \cdots$ by $g(u_1, u_2, \cdots)$, the joint density function of $\bar{x}_0$, $\bar{x}_1$ and $s_0^2$ can be written as

$$g(\bar{x}_0, \bar{x}_1, s_0^2) = g(s_0^2)g(\bar{x}_0, \bar{x}_1/s_0^2),$$

where $g(s_0^2)$ is the density function of $s_0^2$ and $g(\bar{x}_0, \bar{x}_1/s_0^2)$ is the conditional density of $\bar{x}_0$, $\bar{x}_1$ for given $s_0^2$. Since for given $s_0^2$, $\bar{x}_0$ and $\bar{x}_1$ are independently distributed,

$$\begin{aligned} g(\bar{x}_0, \bar{x}_1, s_0^2) &= g(s_0^2)g(\bar{x}_0/s_0^2)g(\bar{x}_1/s_0^2) \\ &= g(\bar{x}_0, s_0^2)g(\bar{x}_1/s_0^2), \end{aligned} \tag{3}$$

where $g(\bar{x}_0, s_0^2)$ is the joint density function of $\bar{x}_0$ and $s_0^2$ and $g(\bar{x}_1/s_0^2)$ is the conditional density function of $\bar{x}_1$ for given $s_0^2$.

Using Gayen's (1949) results for the joint density function of the mean and the variance and the density function of the mean of a random sample drawn from the population (2), the joint distribution of $\bar{x}$ and $s_0^2$ is obtained by substituting $\bar{x}_1 = (n\bar{x} - n_0\bar{x}_0)/n_1$ in (3) and integrating over $\bar{x}_0$, as

$$\begin{aligned} g(\bar{x}, s_0^2) = {} & (n/2\pi)^{\frac{1}{2}}[\nu/\sigma^3 2^{\nu/2}\Gamma(\nu/2)] \exp[-\tfrac{1}{2}(n\zeta^2 + \chi^2)](\chi^2)^{\nu/2-1} \\ & [1 + (n\lambda_3/3!)\{\zeta^3 - 3(n_0/n)\zeta + (3/n)\chi^2\zeta\} \\ & + (n\lambda_4/4!)\{\zeta^4 - 6(n_0/n)\zeta^2 + 3[(n_0/n) + [(1 - 2n_0)/n_0 n^2]n_1] \\ & + (6\chi^2/n)[\zeta^2 - 1 + (n_1/nn_0) + 3\nu\chi^4/nn_0(n_0 + 1)\} \\ & + (n\lambda_3^2/72)\{n\zeta^6 - 3(2n_0 + 3)\zeta^4 + 9(n_0(n_0 + 4)/n)\zeta^2 \\ & - [15(n_0^2/n^2) + (6\nu(\nu - 1)n_1/n_0 n^2)] + (6/n)\chi^2[n\zeta^4 - 3(n_0 + 3)\zeta^2 \\ & + (6(n_0/n) + 3[(\nu - 1)/n_0 n]n_1)] + [9\chi^4/nn_0(n_0 + 1)][n_0(n_0 + 1)\zeta^2 \\ & - 3(n_0 - 1) + [(n_0 + 1)/n]n_1] \\ & + 6(n_0 - 2)\chi^6/nn_0(n_0 + 1)(n_0 + 3)\}] \\ = {} & g_1(\bar{x}, s_0^2, n)\,(\text{say}), \end{aligned} \tag{4}$$

where $\zeta = (\bar{x} - \mu)/\sigma$, $\chi^2 = \nu s_0^2/\sigma^2$.

It may be noted that, $g_1(\bar{x}, s_0^2, n_0) = g_0(\bar{x}_0, s_0^2)$. The joint density function of $\bar{x}$ and $s_0^2$ can therefore be written as,

$$g(\bar{x}, s_0^2) = g_1(\bar{x}, s_0^2, n_0) \quad \text{if} \quad s_0^2 < c,$$
$$= g_1(\bar{x}, s_0^2, n) \quad \text{if} \quad s_0^2 \geqq c.$$

Since $t = (\bar{x} - \mu)/(s_0/n^{\frac{1}{2}}) = \zeta/\chi(\nu n)^{\frac{1}{2}}$, the joint density function of $t$ and $\chi^2$ is obtained by substituting $\zeta = t\chi/(n\nu)^{\frac{1}{2}}$ in (4); which after some rearrangement of terms can be written as

$$h(t, \chi^2) = h_1(t, \chi^2, n_0) \quad \text{if} \quad \chi^2 < \chi_0^2,$$
$$= h_1(t, \chi^2, n) \quad \text{if} \quad \chi^2 \geqq \chi_0^2,$$

where $\chi_0^2 = n_0\nu(z/\sigma^2) = n_0\nu a$ (say),

$$\begin{aligned} h_1(t, \chi^2, n) = {} & [\exp(-T\chi^2/2)(\chi^2)^{\frac{1}{2}(\nu+1)-1}/(2\pi\nu)^{\frac{1}{2}}2^{\nu/2}\Gamma(\nu/2)] \\ & [1 + (\lambda_3/3!)\{[(t^2\chi^2/\nu) + 3(\chi_0^2 - n_0)](t\chi/\nu^{\frac{1}{2}})\}n^{-\frac{1}{2}} \\ & + (\lambda_4/4!)\{[(t^4\chi^4/\nu^2) - 6n_0(t^2\chi^2/\nu) + 3(2n_0 - 1) \\ & + 6\chi^2(t^2\chi^2/\nu - 1)]n^{-1} + (3\nu/n_0)[\nu - 2\chi^2 + \chi^4/(n_0 + 1)]\} \\ & + (\lambda_3^2/72)\{[(t^6\chi^6/\nu^3) - 3(2n_0 + 3)(t^4\chi^4/\nu^2) + 9n_0(n_0 + 4)(t^2\chi^2/\nu) \\ & - 3(3n_0^2 + 6n_0 - 4) + 6\chi^2[(t^4\chi^4/\nu^2) - 3(n_0 + 3)(t^2\chi^2/\nu) \\ & + 3(n_0 + 2)] + 9\chi^4(t^2\chi^2/\nu - 1)]n^{-1} - 6[(n_0 - 2)/n_0][\nu - 3\chi^2 \\ & + [3\chi^4/(n_0 + 1)] - \chi^6/(n_0 + 1)(n_0 + 3)]\}] \end{aligned} \tag{5}$$

and $T = 1 + t^2/\nu$.

The frequency function of $t$ can now be obtained by integrating (5) over $\chi^2$ as

$$p(t) = \int_0^\infty h(t, \chi^2)d\chi^2.$$

Or

$$\begin{aligned} p(t) &= \int_0^{\chi_0^2} h_1(t, \chi^2, n_0)d\chi^2 + \int_{\chi_0^2}^\infty h_1(t, \chi^2, n_0\chi^2/\chi_0^2)d\chi^2 \\ &= \int_0^\infty h_1(t, \chi^2, n_0)d\chi^2 \\ &\quad + \int_{\chi_0^2}^\infty \{h_1(t, \chi^2, n_0\chi^2/\chi_0^2) - h_1(t, \chi^2, n_0)\}d\chi^2 \\ &= p_g(t) + p_s(t) \text{ (say)}, \end{aligned} \tag{6}$$

where

$$p_g(t) = p_0(t) + \lambda_3 p_{\lambda_3}(t) - \lambda_4 p_{\lambda_4}(t) + \lambda_3^2 p_{\lambda_3^2}(t),$$

is Gayen's expression for the density function of Student's $t$ with $\nu$ degrees of freedom,

$$p_0(t) = 1/[\beta(\nu/2, \tfrac{1}{2})\nu^{\frac{1}{2}}T^{(\nu+1)/2}],$$

$$p_{\lambda_3}(t) = (3\nu t - (2\nu + 1)t^3)/6\nu[2\pi(\nu + 1)]^{\frac{1}{2}}T^{(\nu+4)/2}$$

$$\begin{aligned} p_{\lambda_4}(t) = {} & \Gamma(\tfrac{1}{2}(\nu + 3))\{(\nu + 2)t^4 - 6(\nu + 2)t^2 + 3\nu\}/\{24(\pi\nu)^{\frac{1}{2}}(\nu + 1) \\ & \cdot\Gamma(\tfrac{1}{2}(\nu + 4))T^{(\nu+5)/2}\} \end{aligned}$$

$$p_{\lambda_3^2}(t) = \{3\nu^2(2\nu + 13) - 9\nu(\nu + 4)(2\nu + 1)t^2$$
$$-3(\nu + 2)(\nu + 4)(2\nu + 15)t^4 + (\nu + 2)(\nu + 4)(2\nu + 7)t^6\}$$
$$\cdot \Gamma(\tfrac{1}{2}(\nu + 3))/\{144\nu(\nu + 1)(\pi\nu)^{\frac{1}{2}}\Gamma(\tfrac{1}{2}(\nu + 6))T^{(\nu+7)/2}\}$$

and

$$p_s(t) = \lambda_3 p_{\lambda_3 s}(t) + \lambda_4 p_{\lambda_4 s}(t) + \lambda_3^2 p_{\lambda_3^2 s}(t)$$

is the corrective term due to Stein's two sample procedure; $p_{\lambda_3 s}(t)$, $p_{\lambda_4 s}(t)$ and $p_{\lambda_3^2 s}(t)$ are directly obtainable from (5) in terms of incomplete gamma integral of the type

$$\int_{x_0^2}^{\infty} \exp(-T\chi^2/2)(\chi^2)^r d\chi^2 = (2/T)^{r+1}\Gamma(r + 1)\{1 - I(T\chi_0^2/2, r)\},$$

where $I(x, n) = [\Gamma(n + 1)]^{-1}\int_0^x e^{-x}x^n dx$; for example, $p_{\lambda_3 s}(t) = (t/6\nu[2\pi(\nu + 1)]^{\frac{1}{2}} \cdot 2^{\nu/2}\Gamma(\nu/2)\int_{x_0^2}^{\infty}\{[(t^2/\nu) + 3]\chi^2 - 3n_0\}(\chi_0 - \chi)(\chi^2)^{(\nu-1)/2} \exp[-T\chi^2/2]\, d\chi^2$.

It is interesting to note that for any given non-normal situation, as $a = z/\sigma^2$ tends to infinity (i.e. $\chi_0^2 \to \infty$) the frequency function of Stein's $t$ reduces to Gayen's (1949) $p_g(t)$ for Student's $t$.

**4. Tail-area probabilities.** We are often interested in the tail area of a probability distribution and hence consider, $P(t_0) = \int_{-\infty}^{-t_0} p(t)\, dt$ and $P'(t_0) = \int_{t_0}^{\infty} p(t)\, dt$. Integrating (6) $P(t_0)$ is obtained as

$$P(t_0) = P_g(t_0) + P_s(t_0), \tag{7}$$

where

$$P_g(t_0) = P_0(t_0) + \lambda_3 P_{\lambda_3}(t_0) - \lambda_4 P_{\lambda_4}(t_0) + \lambda_3^2 P_{\lambda_3^2}(t_0),$$
$$P_0(t_0) = \int_{-\infty}^{-t_0} p_0(t)dt = \int_{t_0}^{\infty} p_0(t)dt = \tfrac{1}{2}I_{u_0}(\nu/2, \tfrac{1}{2}),$$
$$P_{\lambda_3}(t_0) = \int_{-\infty}^{-t_0} p_{\lambda_3}(t)dt = -\int_{t_0}^{\infty} p_{\lambda_3}(t)dt = (1 + (2\nu + 1)t_0^2/\nu)/6[2\pi(\nu + 1)]^{\frac{1}{2}} \cdot T_0^{(\nu+2)/2},$$
$$P_{\lambda_4}(t_0) = \int_{-\infty}^{-t_0} p_{\lambda_4}(t)dt = \int_{t_0}^{\infty} p_{\lambda_4}(t)dt$$
$$= (\nu/24)I_{u_0}(\nu/2, \tfrac{1}{2}) - [\nu(\nu + 3)/12(\nu + 1)]I_{u_0}[(\nu + 2)/2, \tfrac{1}{2}]$$
$$+ [\nu(\nu + 5)/24(\nu + 1)]I_{u_0}[(\nu + 4)/2, \tfrac{1}{2}],$$
$$P_{\lambda_3^2}(t_0) = \int_{-\infty}^{-t_0} p_{\lambda_3^2}(t)dt = \int_{t_0}^{\infty} p_{\lambda_3^2}(t)dt$$
$$= [\nu(2\nu + 7)/72]I_{u_0}(\nu/2, \tfrac{1}{2}) - [\nu(2\nu^2 + 9\nu + 15)/24(\nu + 1)]$$
$$\cdot I_{u_0}[(\nu + 2)/2, \tfrac{1}{2}] + [\nu(2\nu^2 + 9\nu + 19)/72(\nu + 1)]$$
$$\cdot \{3I_{u_0}[(\nu + 4)/2, \tfrac{1}{2}] - I_{u_0}[(\nu + 6)/2, \tfrac{1}{2}]\},$$

where $u_0 = (T_0)^{-1} = (1 + t_0^2/\nu)^{-1}$ and $I_{u_0}(\nu_1/2, \nu_2/2) = [\beta(\nu_1/2, \nu_2/2)]^{-1}\int_0^{u_0} \cdot u^{(\nu_1/2)-1}(1 - u)^{(\nu_2/2)-1}du$ is the incomplete $\beta$-function; and

$$P_s(t_0) = \lambda_3 P_{\lambda_3 s}(t_0) + \lambda_4 P_{\lambda_4 s}(t_0) + \lambda_3^2 P_{\lambda_3^2 s}(t_0),$$

$$
\begin{aligned}
P_{\lambda_3 s}(t_0) &= \int_{-\infty}^{-t_0} p_{\lambda_3 s}(t)\,dt = -\int_{t_0}^{\infty} p_{\lambda_3 s}(t)\,dt \\
&= \{[(\nu+1)t_0^2/\nu + 2\nu]\chi_0 T_0^{-(\nu+1)/2}/3(\nu-1)(\nu+1)^{\frac{1}{2}}\beta(\nu/2, \tfrac{1}{2})\} \\
&\quad \cdot Q(T_0\chi_0^2, \nu-1) - \{[(2\nu+1)t_0^2/\nu + 1]T_0^{-(\nu+2)/2}/6[2\pi(\nu+1)]^{\frac{1}{2}}\} \\
&\quad \cdot Q(T_0\chi_0^2, \nu),
\end{aligned}
$$

$$
\begin{aligned}
P_{\lambda_4 s}(t_0) &= \int_{-\infty}^{-t_0} p_{\lambda_4 s}(t)\,dt = \int_{t_0}^{\infty} p_{\lambda_4 s}(t)\,dt \\
&= [T_0^{-(\nu+3)/2}/24(\nu+1)\nu^{\frac{1}{2}}\beta(\nu/2, \tfrac{1}{2})][-[T_0\chi_0^2/(\nu-1)]\{(5\nu+4)t_0^3/\nu \\
&\quad + 9t_0\}Q(T_0\chi_0^2, \nu-1) + \{(5\nu+2)t_0^3/\nu - 3t_0\}Q(T_0\chi_0^2, \nu+1)]
\end{aligned}
$$

and

$$
\begin{aligned}
P_{\lambda_3^2 s}(t_0) &= \int_{-\infty}^{-t_0} p_{\lambda_3^2 s}(t)\,dt = \int_{t_0}^{\infty} p_{\lambda_3^2 s}(t)\,dt \\
&= [T_0^{-(\nu+5)/2}/18(\nu+1)\nu^{\frac{1}{2}}\beta(\nu/2, \tfrac{1}{2})][[T_0\chi_0^2/(\nu-1)]\{(\nu^2+8\nu+6) \\
&\quad \cdot t_0^5/\nu^2 + 2(4\nu+11)t_0^3/\nu + 15\,t_0\}Q(T_0\chi_0^2, \nu-1) - \{(\nu^2+6\nu \\
&\quad +2)t_0^5/\nu^2 + 2(\nu-2)t_0^3/\nu - 3t_0\}Q(T_0\chi_0^2, \nu+1) - t_0(t_0^2/\nu+3)^2 \\
&\quad \cdot [\exp(-T_0\chi_0^2/2)](T_0\chi_0^2/2)^{(\nu+1)/2}/\Gamma(\tfrac{1}{2}(\nu+1))]
\end{aligned}
$$

where $Q(\chi_0^2, \nu) = [2^{\nu/2}\Gamma(\nu/2)]^{-1}\int_{x_0^2}^{\infty} e^{-x/2}x^{\nu/2-1}dx$ is the complement of the probability integral of chi-square with $\nu$ degrees of freedom.

Similarly

$$P'(t_0) = P_g'(t_0) + P_s'(t_0), \tag{8}$$

where

$$
\begin{aligned}
P_g'(t_0) &= P_0(t_0) - \lambda_3 P_{\lambda_3}(t_0) - \lambda_4 P_{\lambda_4}(t_0) + \lambda_3^2 P_{\lambda_3^2}(t_0), \\
P_s'(t_0) &= -\lambda_3 P_{\lambda_3 s}(t_0) + \lambda_4 P_{\lambda_4 s}(t_0) + \lambda_3^2 P_{\lambda_3^2 s}(t_0).
\end{aligned}
$$

If two tails are considered, then

$$P_{t_0} = \text{Prob.}\{|t| > t_0\} = P(t_0) + P'(t_0). \tag{9}$$

For $\lambda_3 = \pm 1.0, \pm 0.6, \pm 0.4$ ($\lambda_4 = 0$) and $\lambda_4 = -0.5, 1.0, 2.0, 2.5$ ($\lambda_3 = 0$), the values of the lower tail area at the lower 2.5% and 0.5% points of the normal theory $t_\nu$ for $\nu = 4, 9$ and $a = 1.0, 0.1, 0.01, 0.001$ are shown in Table 1(a) and Table 1(b). The values of $P_{t_0}$ at 5% and 1% points of the normal theory $t_\nu$ can be obtained from the same table. It will be observed that the tail area varies considerably with $a$ and hence with the unknown variance $\sigma^2$. Even for near normal populations the variation is noticeable, e.g. when $\lambda_3 = 0.4$, $\lambda_4 = -0.5$, $\nu = 4$; $P(2.776) = 0.038$ for $a = 0.1$ and $P(2.776) = 0.024$ for $a = 0.001$. For higher values of $\lambda_3$ and $\lambda_4$ the tail area changes rapidly with $a$, e.g. when $\lambda_3 = 1.0$, $\lambda_4 = -0.5$, $\nu = 9$; $P(2.262) = 0.0179$ for $a = 0.1$ and $P(2.262) = 0.0041$ for $a = 0.001$.

The effect of $\lambda_4$ though small on Student's $t$ (as observed by Geary (1935)

TABLE 1(a)

*Values of the lower tail area at the lower 2.5% and 0.5% points of the normal theory $t_\nu$ for asymmetrical populations with $\lambda_4 = 0$*

| $a$ \ $n_0$ | $\lambda_3$: −1.0 | | −0.6 | | −0.4 | | 0 | 0.4 | | 0.6 | | 1.0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 5 | 10 | 5 | 10 | 5 | 10 | | 5 | 10 | 5 | 10 | 5 | 10 |
| 1.0 | 0.0162 | 0.0133 | 0.0166 | 0.0158 | 0.0184 | 0.0181 | | 0.0358 | 0.0348 | 0.0427 | 0.0408 | 0.0597 | 0.0550 |
| 0.1 | 0.0166 | 0.0129 | 0.0167 | 0.0155 | 0.0183 | 0.0180 | 0.0250 | 0.0361 | 0.0350 | 0.0433 | 0.0411 | 0.0609 | 0.0556 |
| 0.01 | 0.0158 | 0.0165 | 0.0186 | 0.0191 | 0.0205 | 0.0208 | | 0.0306 | 0.0302 | 0.0339 | 0.0332 | 0.0411 | 0.0400 |
| 0.001 | 0.0196 | 0.0240 | 0.0221 | 0.0246 | 0.0231 | 0.0248 | | 0.0265 | 0.0250 | 0.0271 | 0.0250 | 0.0279 | 0.0247 |
| | | | | | | | | | | | | | |
| 1.0 | 0.0044 | 0.0037 | 0.0034 | 0.0029 | 0.0035 | 0.0032 | | 0.0081 | 0.0086 | 0.0103 | 0.0111 | 0.0160 | 0.0174 |
| 0.1 | 0.0044 | 0.0037 | 0.0034 | 0.0029 | 0.0035 | 0.0031 | 0.0050 | 0.0081 | 0.0086 | 0.0103 | 0.0111 | 0.0160 | 0.0174 |
| 0.01 | 0.0031 | 0.0028 | 0.0033 | 0.0032 | 0.0037 | 0.0037 | | 0.0071 | 0.0070 | 0.0084 | 0.0082 | 0.0117 | 0.0111 |
| 0.001 | 0.0032 | 0.0045 | 0.0040 | 0.0048 | 0.0044 | 0.0049 | | 0.0056 | 0.0050 | 0.0058 | 0.0050 | 0.0062 | 0.0049 |

TABLE 1(b)

*Values of the lower tail area at the lower 2.5% and 0.5% points of the normal theory $t_\nu$ for some symmetrical non-normal populations*

| $a$ \ $n_0$ | $\lambda_4$: −0.5 | | 0 | 1.0 | | 2.0 | | 2.5 | |
|---|---|---|---|---|---|---|---|---|---|
| | 5 | 10 | | 5 | 10 | 5 | 10 | 5 | 10 |
| 1.0 | 0.0265 | 0.0257 | | 0.0221 | 0.0236 | 0.0192 | 0.0221 | 0.0178 | 0.0214 |
| 0.1 | 0.0265 | 0.0259 | 0.0250 | 0.0221 | 0.0233 | 0.0192 | 0.0216 | 0.0177 | 0.0207 |
| 0.01 | 0.0235 | 0.0236 | | 0.0280 | 0.0278 | 0.0311 | 0.0306 | 0.0326 | 0.0320 |
| 0.001 | 0.0226 | 0.0229 | | 0.0299 | 0.0292 | 0.0347 | 0.0335 | 0.0372 | 0.0356 |
| | | | | | | | | | |
| 1.0 | 0.0056 | 0.0054 | | 0.0039 | 0.0041 | 0.0028 | 0.0032 | 0.0023 | 0.0028 |
| 0.1 | 0.0056 | 0.0055 | 0.0050 | 0.0039 | 0.0041 | 0.0028 | 0.0032 | 0.0022 | 0.0027 |
| 0.01 | 0.0048 | 0.0046 | | 0.0054 | 0.0059 | 0.0058 | 0.0068 | 0.0059 | 0.0072 |
| 0.001 | 0.0043 | 0.0042 | | 0.0064 | 0.0066 | 0.0077 | 0.0082 | 0.0084 | 0.0090 |

TABLE 2

*Values of the power function of Stein's two-sided t-test at 5% level with $n_0 = 5$ for different non-normal situations*

| $\delta$ | $a$ | $(\lambda_3, \lambda_4)$ (0, 0) | (−1, 0) | (+1, 0) | (−0.6, 0) | (+0.6, 0) | (0, −1) | (0, 2) | (0.2.5) | (−0.6, 2) | (+0.6, 2) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1.0 | | 0.0760 | 0.0760 | 0.0594 | 0.0594 | 0.0558 | 0.0385 | 0.0356 | 0.0478 | 0.0478 |
| | 0.1 | 0.0500 | 0.0776 | 0.0776 | 0.0600 | 0.0600 | 0.0559 | 0.0383 | 0.0354 | 0.0482 | 0.0482 |
| | 0.01 | | 0.5700 | 0.0570 | 0.0525 | 0.0525 | 0.0440 | 0.0621 | 0.0651 | 0.0646 | 0.0646 |
| | 0.001 | | 0.0476 | 0.0476 | 0.0490 | 0.0490 | 0.0403 | 0.0694 | 0.0743 | 0.0685 | 0.0685 |
| 0.5 | 1.0 | | 0.0665 | 0.0988 | 0.0571 | 0.0765 | 0.0633 | 0.0471 | 0.0444 | 0.0463 | 0.0657 |
| | 0.1 | | 0.0680 | 0.1022 | 0.0574 | 0.0780 | 0.0636 | 0.0466 | 0.0437 | 0.0461 | 0.0666 |
| | 0.01 | 0.0579 | 0.0568 | 0.0724 | 0.0556 | 0.0650 | 0.0515 | 0.0707 | 0.0739 | 0.0685 | 0.0778 |
| | 0.001 | | 0.0579 | 0.0629 | 0.0555 | 0.0585 | 0.0479 | 0.0780 | 0.0830 | 0.0756 | 0.0786 |
| 1.0 | 1.0 | | 0.0732 | 0.1367 | 0.0731 | 0.1112 | 0.0888 | 0.0772 | 0.0753 | 0.0654 | 0.1034 |
| | 0.1 | | 0.0756 | 0.1453 | 0.0732 | 0.1150 | 0.0900 | 0.0749 | 0.0724 | 0.0639 | 0.1050 |
| | 0.01 | 0.0850 | 0.0754 | 0.1068 | 0.0778 | 0.0966 | 0.0779 | 0.0991 | 0.1026 | 0.0919 | 0.1107 |
| | 0.001 | | 0.0776 | 0.0875 | 0.0811 | 0.0871 | 0.0743 | 0.1061 | 0.1114 | 0.1023 | 0.1082 |
| 1.5 | 1.0 | | 0.1102 | 0.1952 | 0.1203 | 0.1713 | 0.1422 | 0.1415 | 0.1414 | 0.1199 | 0.1709 |
| | 0.1 | | 0.1117 | 0.2144 | 0.1188 | 0.1803 | 0.1459 | 0.1341 | 0.1321 | 0.1109 | 0.1725 |
| | 0.01 | 0.1420 | 0.1245 | 0.1703 | 0.1302 | 0.1576 | 0.1351 | 0.1556 | 0.1590 | 0.1439 | 0.1713 |
| | 0.001 | | 0.1329 | 0.1475 | 0.1370 | 0.1457 | 0.1322 | 0.1616 | 0.1665 | 0.1566 | 0.1653 |
| 2.0 | 1.0 | | 0.2053 | 0.2869 | 0.2209 | 0.2698 | 0.2408 | 0.2534 | 0.2555 | 0.2293 | 0.2780 |
| | 0.1 | 0.2450 | 0.1969 | 0.3207 | 0.2128 | 0.2871 | 0.2478 | 0.2394 | 0.2380 | 0.2072 | 0.2815 |
| | 0.01 | | 0.2215 | 0.2771 | 0.2298 | 0.2632 | 0.2408 | 0.2532 | 0.2553 | 0.2381 | 0.2714 |
| | 0.001 | | 0.2352 | 0.2528 | 0.2393 | 0.2499 | 0.2388 | 0.2573 | 0.2603 | 0.2516 | 0.2622 |
| 2.5 | 1.0 | | 0.3728 | 0.4309 | 0.3843 | 0.4192 | 0.3986 | 0.4080 | 0.4095 | 0.3906 | 0.4254 |
| | 0.1 | | 0.3436 | 0.4728 | 0.3653 | 0.4428 | 0.4033 | 0.3986 | 0.3978 | 0.3622 | 0.4397 |
| | 0.01 | 0.4017 | 0.3748 | 0.4340 | 0.3849 | 0.4205 | 0.4003 | 0.4045 | 0.4052 | 0.3877 | 0.4232 |
| | 0 001 | | 0.3920 | 0.4107 | 0.3960 | 0.4072 | 0.3991 | 0.4070 | 0.4083 | 0.4012 | 0.4124 |

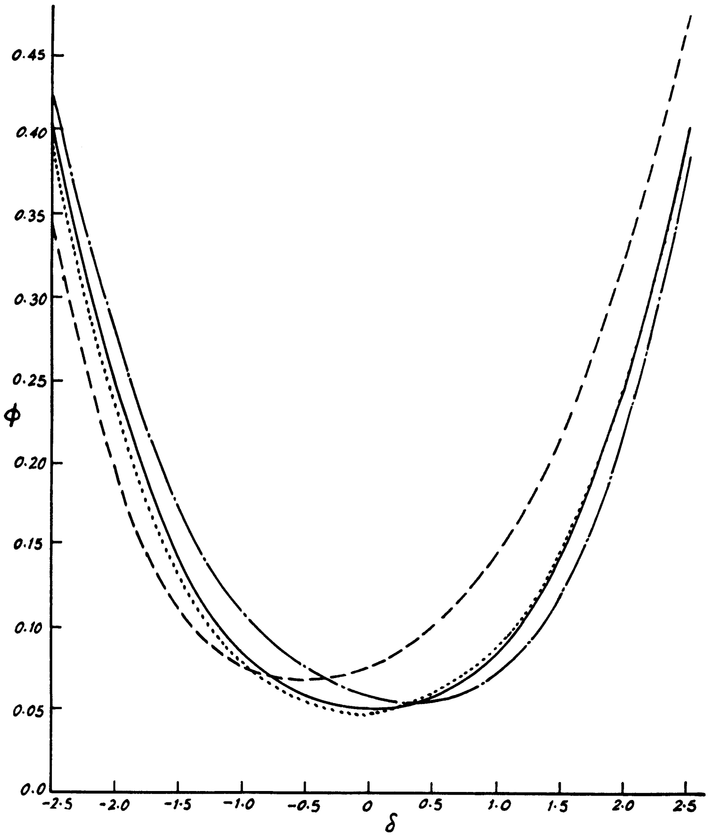



FIG. 1(a). Power curves of Stein's two-sided $t$ test with $n_0 = 5$, $\alpha = 0.05$ in the case of asymmetrical populations with $\lambda_4 = 0$

————————— Normal
——·——·—— $\lambda_3 = -0.6$, $a = 1.0$
----------------- $\lambda_3 = 1.0$, $a = 0.1$
·············· $\lambda_3 = 1.0$, $a = 0.001$

and Gayen (1949)) may be somewhat more on Stein's $t$, e.g. when $\lambda_3 = 0$, $\lambda_4 = 2.5$, $\nu = 4$, $a = 0.001$ the estimate of the two-sided tail area at the 5% point of the normal theory $t$ is 0.074 whereas the corresponding value for the Student's $t$ is only 0.035.

The effect of $\lambda_3$ on Stein's $t$ is considerable, as it is on Student's $t$. The negative values of $\lambda_3$ tend to decrease the lower tail area but simultaneously increase the area of the upper tail. The reverse is the case for positive values of $\lambda_3$. If two tails are considered, the corrective terms due to $\lambda_3$ will vanish but those due to $\lambda_3^2$ will not.

**5. Stein's two sample scheme for non-normal populations.** If Stein's two sample $t$ test of Section 2 is used for testing the mean of a non-normal popula-

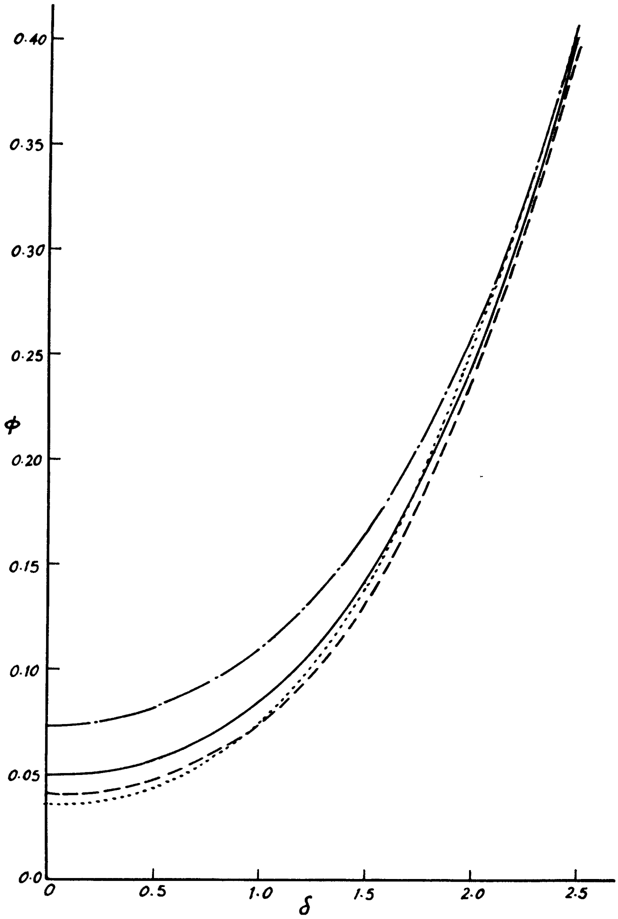



FIG. 1(b). Power curves of Stein's two-sided $t$ test with $n_0 = 5$, $\alpha = 0.05$ in the case of symmetrical non-normal populations

———————— Normal
—·—·—·— $\lambda_4 = 2.5$, $\alpha = 0.001$
·········· $\lambda_4 = 2.5$, $\alpha = 1.0$
-------------- $\lambda_4 = -1.0$, $\alpha = 0.001$

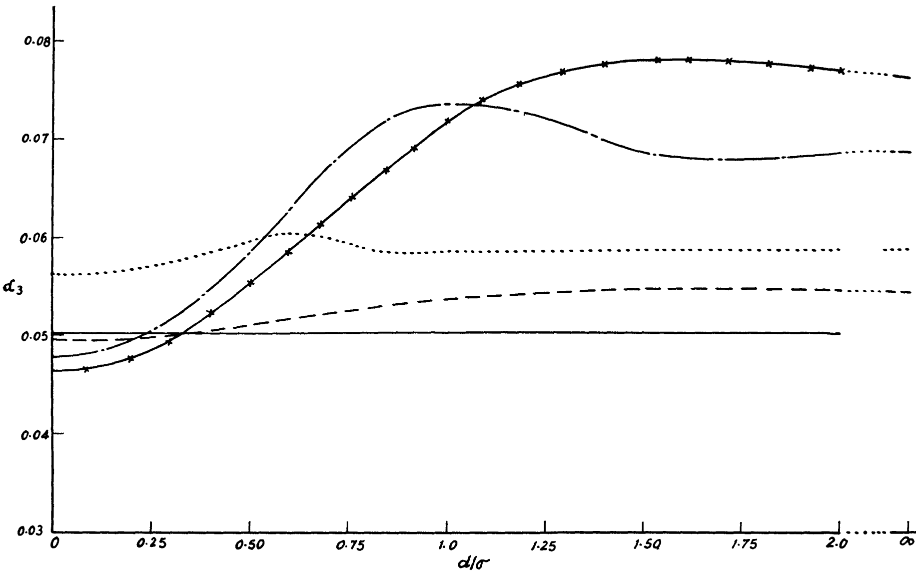
tion, the preassigned value of $\alpha$ is not attained and the actual error of the first kind is given by $\alpha_1 = P\{t(\alpha/2, \nu)\} + P'\{t(\alpha/2, \nu\}$.

The power of the test

$$\phi_1(\mu) = 1 - \beta_1(\mu) = P\{t(\alpha/2, \nu) - \delta\} + P'\{t(\alpha/2, \nu) + \delta\},$$

where $\delta = (\mu_0 - \mu)/z^{\frac{1}{2}}$ no longer remains independent of the unknown variance $\sigma^2$.

For testing $H_0(\mu = \mu_0)$ against one sided alternatives $\mu > \mu_0$, the actual error



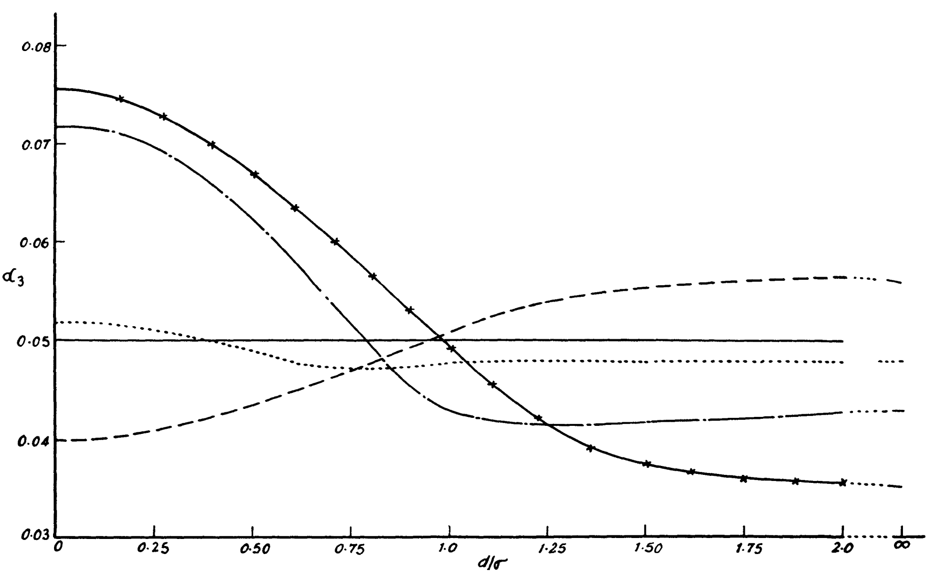




FIG. 2(a). Relationship between $\alpha_3(\alpha = 0.05)$ and $d/\sigma$ in the case of asymmetrical populations with $\lambda_4 = 0$

——— Normal
—✳—✳— $n_0 = 5, \lambda_3 = \pm 1$
—·—·— $n_0 = 10, \lambda_3 = \pm 1$
- - - - - $n_0 = 5, \lambda_3 = \pm .4$
· · · · · $n_0 = 25, \lambda_3 = \pm 1$

FIG. 2(b) Relationship between $\alpha_3(\alpha = 0.05)$ and $d/\sigma$ in the case of symmetrical non-normal populations

——— Normal
—✳—✳— $n_0 = 5, \lambda_4 = 2.5$
—·—·— $n_0 = 10, \lambda_4 = 2.5$
- - - - - $n_0 = 5, \lambda_4 = -1.0$
· · · · · $n_0 = 25, \lambda_4 = 2.5$

of the first kind is $\alpha_2 = P'\{t(\alpha, \nu)\}$ and the power function is $\phi_2(\mu) = P'\{t(\alpha, \nu) + \delta\}$.

For certain values of $\lambda_3$, $\lambda_4$ and $a(=z/\sigma^2)$ the values of the power function for the two-sided Stein's test for an initial sample of size 5 at 5 percent level of significance are shown in Table 2. The effect of skewness and kurtosis of the parent population on the power function is illustrated in figures 1(a) and (b). The power of Stein's test though desired to be independent of $\sigma^2$ is seen to depend considerably on the unknown variance. For very small values of $\lambda_3$ and $\lambda_4$ the variation of the power curve with $\sigma^2$ is not much. The asymmetry of the parent population tend to increase the power in one side of $\mu_0$ but decrease it on the other. For symmetrical non-normal populations, the power curve is symmetrical with respect to $\mu = \mu_0$ and the increase or the decrease of the power function in the neighbourhood of $\mu_0$ depends largely on the unknown variance.

The departure from normality of the parent population also affects the preassigned value of the confidence level $\alpha$, of Stein's fixed size confidence interval. The actual value of the confidence level is

$$\alpha_3 = P\{t(\alpha/2, \nu)\} + P'\{t(\alpha/2, \nu)\}.$$

The true confidence level $\alpha_3$ is seen to depend considerably on the ratio of the size of the interval to the population standard deviation particularly when the size $n_0$ of the initial sample is small. The relationship between $\alpha_3$ and $d/\sigma$ for asymmetrical and symmetrical non-normal parent populations is illustrated in figures 2(a) and 2(b) respectively. The effect of skewness on the confidence level appears to be not much when the pre-assigned length is small, but as the pre-assigned length is increased, $\alpha_3$ increases and for highly skewed populations (say $\lambda_3^2 = 1$) the effect may be serious. For extreme (very high or very low) value of $d/\sigma$ the effect of kurtosis may be serious. Positive values of $\lambda_4$ tend to increase $\alpha_3$ for low $d/\sigma$ and decrease it for high values of $d/\sigma$. The reverse happens when $\lambda_4$ is negative.

**6. Summary and conclusions.** The distribution of Stein's $t$ in non-normal samples has been derived with reference to the parent population specified by the first four terms of an Edgeworth series. It contains in addition to the frequency function of Student's $t$ for non-normal parent, the corrective terms due to $\lambda_3$, $\lambda_4$ and $\lambda_3^2$. The validity of Stein's two sample schemes in a non-normal situation for testing the mean or for estimating it by a confidence interval of pre-assigned length are then examined by obtaining the corrected power function and the confidence level respectively.

The study on the whole, shows that Stein's $t$ is more sensitive to non-normality of the parent populations than Student's $t$. If the parent population is not normal, the power function of Stein's test, though desired to be independent of $\sigma^2$, is found to depend considerably on the unknown variance.

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