LOCAL CONVERGENCE OF MARTINGALES AND THE LAW OF LARGE NUMBERS

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0. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and $\mathfrak{F}_n$ be increasing Borel subfields of $\mathfrak{F}$. $(y_n, \mathfrak{F}_n, n \geqq 1)$ is said to be a stochastic sequence if y_n is extended real valued and $\mathfrak{F}_n$-measurable for each n. A stochastic sequence $(y_n, \mathfrak{F}_n, n \geqq 1)$ is called a submartingale (or martingale), if $E(y_n)$ exists (that is $Ey^+ < \infty$ or $Ey^- < \infty$) and $E(y_{n+1} \mid \mathfrak{F}_n) \geqq y_n$ (or $E(y_{n+1} \mid \mathfrak{F}_n) = y_n$) a.e. for each n. A stopping variable t is an extended positive integer valued random variable such that the set $[t = n] \varepsilon \mathfrak{F}_n$ for each positive integer n. For an extended real number a, define $a^+ = \max(0, a)$ and $a^- = \max(0, -a)$. For a set A, $I(A)$ denotes the characteristic function of the set A.

Recently, Neveu ([8], p. 143) proves a new submartingale convergence theorem, namely if $(s_n, \mathfrak{F}_n, n \geqq 1)$ is a submartingale with $E(s_n^+) < \infty$, then s_n has a limit a.e. where $\sum_2^\infty (E(s_n^+ \mid \mathfrak{F}_{n-1}) - s_{n-1}^+) < \infty$. Neveu's result suggests the present paper. In Section 1 we will generalize his result and in Section 2 prove a local convergence theorem of martingales, which extends a result of Loève [7] and improves a result of Lévy-Doob ([4] p. 320). Section 3 is denoted to the law of large number and a result due to Lévy-Neveu ([5]; [8], p. 141) is extended in this section.

1. Local convergence of submartingales.

THEOREM 1. *Let $(s_n, \mathfrak{F}_n, n \geqq 1)$ be a submartingale with $E(s_1^-) < \infty$, and $(z_n, \mathfrak{F}_{n-1}, n \geqq 2)$ and $(y_n, \mathfrak{F}_n, n \geqq 2)$ be two stochastic sequences such that y_n is finite valued for each n. Let $z_1 = y_1 = 0$ and*

(1) $$s_n \leqq z_n + y_n, \qquad n \geqq 2.$$

For $b > 0$, let t be the first n such that $y_n > b$ and let

(2) $$E(y_t I[t < \infty]) < \infty.$$

Then s_n converges a.e. *where*

(3) $$\sup z_n < \infty, \qquad \sup y_n < b.$$

PROOF. For any fixed $a > 0$, let t' be the first n such that $z_n > a$ and $w_n = \min(t' - 1, t, n)$. Then w_n is a sequence of bounded stopping variables and $w_n \leqq w_{n+1}$. Hence ([4], *p.* 303) s_{w_n} is a submartingale, since $E(s_1^-) < \infty$ implies that $E(s_n^-) < \infty$ for each n. Now

$$E(s_{w_n}^+) \leqq E(z_{w_n}^+) + \sum_{j=1}^n E(y_j^+ I[w_n = j])$$

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$$\leqq a + \sum_1^n E(y_j^+(I[t' - 1 \geqq j = t] + I[t' - 1 = j < t])) + E(y_n^+ I[t' - 1 > n < t])$$
$$\leqq a + b + \sum_1^n E(y_j^+ I[t' - 1 \geqq j = t]) \leqq a + b + E(y_t^+ I[t < \infty]).$$

By the standard martingale convergence theorem of Doob ([6], p. 393), s_{w_n} converges a.e. Hence s_n converges a.e. where $t = t' = \infty$, i.e., a.e. where $\sup z_n < a$ and $\sup y_n < b$. Since a is arbitrary, the proof is completed.

When $z_n = 0$ for each n, Theorem 1 reduces to a result of ([3], Corollary 6(i)), which in turn implies the standard martingale convergence theorems of Doob and Snell ([3], p. 344).

Corollary 1. *The condition* (2) *in Theorem* 1 *can be replaced by*

$$(2') \qquad E(\sup_{n \geqq 1} (y_{n+1} - y_n^+)) < \infty.$$

Proof. Let t be defined as in Theorem 1. Then $t \geqq 2$ and (2′) implies that $E(y_t I[t < \infty]) \leqq E(\sup (y_{n+1} - y_n^+)) + E(y_{t-1}^+ I[t < \infty]) < \infty$. Hence (2′) implies (2).

When $z_n = s_{n-1}$, $y_n = s_n - s_{n-1}$, and $E(\sup y_n) < \infty$, Corollary 1 reduces to a result of Doob ([4], p. 320).

Corollary 2. *Let* $(s_n, \mathfrak{F}_n, n \geqq 1)$ *be a submartingale, and let* $(y_n, \mathfrak{F}_n, n \geqq 1)$ *be a stochastic sequence such that* $E|y_n| < \infty$, $E(\sup (y_{n+1}^- - y_n^-)) < \infty$, *and* $s_n \leqq y_n$. *Then* s_n *converges* a.e. *where*

$$(4) \qquad \sup \sum_1^m (E(y_{n+1} \mid \mathfrak{F}_n) - y_n) < \infty,$$

$$(5) \qquad \sup y_n^- < \infty.$$

Proof. Put $z_m = \sum_2^m (E(y_n \mid \mathfrak{F}_{n-1}) - y_n)$. Then $(z_n, \mathfrak{F}_n, n \geqq 2)$ is a martingale, and

$$z_m = -y_m + \sum_3^m (E(y_n \mid \mathfrak{F}_{n-1}) - y_{n-1}) + E(y_2 \mid \mathfrak{F}_1)$$
$$\leqq y_m^- + \sum_3^m (E(y_n \mid \mathfrak{F}_{n-1}) - y_{n-1}) + E(y_2 \mid \mathfrak{F}_1) = y_m^- + u_m,$$

say. Then u_m is $\mathfrak{F}_{m-1}$-measurable for $m \geqq 2$. By Corollary 1, z_m converges a.e. where $\sup_{n \geqq 2} y_n < \infty$ and $\sup_{n \geqq 2} u_n < \infty$. Since $z_n + s_n$ is a submartingale and $z_n + s_n \leqq z_n + y_n \leqq u_n$, $z_n + s_n$ converges a.e. where $\sup u_n < \infty$. Hence s_n converges a.e. where (4) and (5) hold.

Corollary 3. *Let* $(s_n, \mathfrak{F}_n, n \geqq 1)$ *be a submartingale and* $p \geqq 1$.

(a) *If* $E(s_n^+)^p < \infty$, *then* s_n *converges* a.e. *where*

$$(6) \qquad \sum_2^\infty (E((s_n^+)^p \mid \mathfrak{F}_{n-1}) - (s_{n-1}^+)^p) < \infty.$$

(b) *If* $E|s_n|^p < \infty$, *then* s_n *converges* a.e. *where*

$$(7) \qquad \sum_2^\infty (E(|s_n|^p \mid \mathfrak{F}_{n-1}) - |s_{n-1}|^p) < \infty.$$

Proof. Since $s_n \leqq s_n^+ \leqq (s_n^+)^p + 1$ and $s_n \leqq |s_n| \leqq |s_n|^p + 1$, Corollary 3 follows immediately from Corollary 2.

For $p = 1$ or $p > 1$ and s_n being non-negative, Corollary 3(a) has been recently proved by Neveu ([8], p. 143).

COROLLARY 4. *Let* $(y_n, \mathfrak{F}_n, n \geqq 1)$ *be a stochastic sequence such that* $E|y_n| < \infty$ *and* $E(\sup (y_{n+1}^- - y_n^-)) < \infty$. *Then* y_n *converges* a.e. *where*

$$\text{(8)} \qquad \sup y_n^- < \infty, \qquad \sum_1^\infty (E(y_{n+1} \mid \mathfrak{F}_n) - y_n) \text{ converges.}$$

PROOF. Put z_n and u_n as in the proof of Corollary 2. Then z_n converges a.e. where $\sup y_n^- < \infty$ and $\sup u_n < \infty$. Since $y_n = u_n - z_n$, y_n converges a.e. where (8) holds.

2. Martingale convergence theorems. In this and the next section, we will assume that $(s_n, \mathfrak{F}_n, n \geqq 1)$ is a fixed martingale with $E|s_n| < \infty$ and $x_1 = s_1$, $x_n = s_n - s_{n-1}$ for $n \geqq 2$.

THEOREM 2. *Let* A *be the set where*

$$\text{(9)} \qquad \sum_2^\infty E(|x_n|^2 I[|x_n| \leqq a_n] + |x_n| I[|x_n| > a_n] \mid \mathfrak{F}_{n-1}) < \infty,$$

for some constants $a_n \geqq c > 0$. *Then* s_n *converges* a.e. *in* A.

PROOF. Put $x'_n = x_n I[|x_n| \leqq a_n]$. By the martingale property,

$$|E(x'_n \mid \mathfrak{F}_{n-1})| = |E(x_n I[|x_n| > a_n] \mid \mathfrak{F}_{n-1})| \leqq E(|x_n| I[|x_n| > a_n] \mid \mathfrak{F}_{n-1})$$

a.e. Then

$$\text{(10)} \qquad \sum_2^\infty |E(x'_n \mid \mathfrak{F}_{n-1})| < \infty \qquad \text{a.e. in } A.$$

Since

$$\sum_2^\infty P(x'_n \neq x_n \mid \mathfrak{F}_{n-1}) = \sum_2^\infty P(|x_n| > a_n \mid \mathfrak{F}_{n-1})$$
$$\leqq c^{-1} \sum_2^\infty E(|x_n| I[|x_n| > a_n] \mid \mathfrak{F}_{n-1}) < \infty, \quad \text{a.e. in } A,$$

by Corollary 3 or a theorem of Lévy ([5], p. 247 or [4], p. 324), we have $\sum_2^\infty I[x_n \neq x'_n] < \infty$ a.e. in A. It follows that

$$\text{(11)} \qquad P(A[x_n \neq x'_n, \text{ i.o.}]) = 0.$$

Put $y_n = x'_1 + \cdots + x'_n - E(x'_2 \mid \mathfrak{F}_1) - \cdots - E(x'_n \mid \mathfrak{F}_{n-1})$. Then $(y_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale, and

$$E(y_n^2 \mid \mathfrak{F}_{n-1}) - y_{n-1}^2 = E((x'_n)^2 \mid \mathfrak{F}_{n-1}) - E^2(x_n \mid \mathfrak{F}_{n-1})$$
$$\leqq E((x'_n)^2 \mid \mathfrak{F}_{n-1}) \leqq E(x_n^2 I[|x_n| \leqq a_n] \mid \mathfrak{F}_{n-1}).$$

Hence,

$$\text{(12)} \qquad \sum_2^\infty (E(y_n^2 \mid \mathfrak{F}_{n-1}) - y_{n-1}^2) < \infty \qquad \text{a.e. in } A.$$

By Corollary 3(b), y_n converges a.e. in A. From (10) and (11), we have that s_n converges a.e. in A.

Under the condition that $\sum_2^\infty E(x_n^2 I[|x_n| \leqq a_n] + |x_n| I[|x_n| > a_n]) < \infty$, Theorem 2 has been proved by Loève ([7], p. 286).

COROLLARY 5. *Let* $1 \leqq p \leqq 2$. *Then* s_n *converges* a.e. *where*

$$\text{(13)} \qquad \sum_2^\infty E(|x_n|^p \mid \mathfrak{F}_{n-1}) < \infty.$$

PROOF. Since

$\sum_2^\infty E(|x|^2 I[|x_n| \leqq 1] + |x_n| I[|x_n| > 1] \mid \mathfrak{F}_{n-1}) \leqq \sum_2^\infty E(|x_n|^p \mid \mathfrak{F}_{n-1}) < \infty$,

Corollary 5 follows immediately from Theorem 2.

Under the condition that $x_1, x_2, \cdots$ are independent, Corollary 5 has been proved by Marcinkiewicz and Zygmund ([9], p. 74). When $p = 2$, Corollary 5 follows from Corollary 3(b) and it improves a result of Lévy-Doob ([4], p. 320) by removing one of their conditions for the convergence of x_n.

THEOREM 3. *Let $(z_n, \mathfrak{F}_n, n \geqq 1)$ be a strictly positive stochastic sequence and $p > 2$. Let B be the set such that*

$$(14) \qquad \sum_1^\infty z_n < \infty, \qquad \sum_2^\infty E(|x_n|^p \mid \mathfrak{F}_{n-1}) z_n^{1-(p/2)} < \infty.$$

Then $\sum_2^\infty E(x_n^2 \mid \mathfrak{F}_{n-1}) < \infty$ a.e. in B, and therefore s_n converges a.e. in B.

PROOF. If $E^{2/p}(|x_n|^p \mid \mathfrak{F}_{n-1}) > z_n$, then, since $p > 2$,

$$E^{2/p}(|x_n|^p \mid \mathfrak{F}_{n-1}) = E(|x_n|^p \mid \mathfrak{F}_{n-1}) E^{(2/p)-1}(|x_n|^p \mid \mathfrak{F}_{n-1}) \leqq z_n^{1-(p/2)} E(|x_n|^p \mid \mathfrak{F}_{n-1}).$$

Hence

$$E(x_n^2 \mid \mathfrak{F}_{n-1}) \leqq E^{2/p}(|x_n|^p \mid \mathfrak{F}_{n-1}) \leqq \max [z_n, z_n^{1-(p/2)} E(|x_n|^p \mid \mathfrak{F}_{n-1})].$$

Therefore, $\sum_2^\infty E(x_n^2 \mid \mathfrak{F}_{n-1}) < \infty$ a.e. in B, and it follows from Corollary 5 that s_n converges a.e. in B.

COROLLARY 6. *Let $p > 2$ and δ_n be a sequence of positive number such that*

$$(15) \qquad \sum_1^\infty \delta_n < \infty, \qquad \sum_1^\infty E|x_n|^p \delta_n^{1-(p/2)} < \infty.$$

Then s_n converges a.e. In particular, if

$$(16) \qquad \sum_1^\infty E|x_n|^p [n(\log n)^{1+\epsilon}]^{(p/2)-1} < \infty$$

for some $\epsilon > 0$, then s_n converges a.e.

Corollary 6 follows immediately from Theorem 3. The fact that the factor $\log n$ in (16) cannot be dropped is shown by the following example. Let $z_1, z_2, \cdots$ be independent with $P(z_n = 0) = P(z_n = 1) = \frac{1}{2}$ and set

$$(17) \qquad x_n = (-1)^{z_n}(n \log n)^{-\frac{1}{2}}, \qquad n \geqq 2.$$

Then $x_2, x_3, \cdots$ are independent with $Ex_n = 0$ and $\sum_2^\infty E|x_n|^2 = \sum_2^\infty 1/n \log n = \infty$. From a result of Doob ([4], p. 339), it follows that $s_n = x_2 + \cdots + x_n$ diverges a.e., but

$$(18) \qquad \sum_2^\infty E|x_n|^3 n^{\frac{1}{2}} < \infty.$$

Both Corollary 6 and the preceding remark are results of an unpublished paper by Chow, Mallows, and Robbins [2].

THEOREM 4. *Let $z, z_1, z_2, \cdots$ be independent, identically distributed with $E(z) = 0$, and $\mathfrak{A}_n$ be the Borel field generated by z_n. Then*

$$(19) \qquad E(|z| \log^+ |z|) < \infty,$$

if, and only if, for every sequence $\mathcal{B}_n$ *of Borel fields such that* $\mathcal{B}_n \subset \mathcal{A}_n$, *we have*

$$P\,[\textstyle\sum_{n=1}^{\infty} n^{-1}E(z_n \mid \mathcal{B}_n) \quad \text{converges}] = 1. \tag{20}$$

PROOF. Let (19) hold. Define $f_n(t) = t^2/n^2$ for $|t| \leqq n$ and $f(t) = 2|t|/n - 1$ for $n \leqq |t|$. Then f_n is convex and monotonically increasing for $0 \leqq t < \infty$. Put $y_n = E(z_n \mid \mathcal{B}_n)$. Then $E(f_n(z_n) \mid \mathcal{B}_n) \geqq f_n[E(z_n \mid \mathcal{B}_n)] = f_n(y_n)$. Hence $Ef_n(z_n) \geqq Ef_n(y_n)$,

$$\textstyle\sum_1^{\infty} E(n^{-2}y_n^{\,2}I[|y_n| \leqq n] + |n^{-1}y_n|I[|y_n| > n]) \leqq \sum_1^{\infty} Ef_n(y_n) \leqq \sum_1^{\infty} Ef_n(z_n).$$

In ([9], pp. 77–78), it has been proved that (19) implies that

$$\textstyle\sum_1^{\infty} E(n^{-2}z_n^{\,2}I[|z_n| \leqq n] + |n^{-1}z_n|I[|z_n| > n]) < \infty. \tag{21}$$

Hence, $\sum_1^{\infty} Ef_n(z_n) < \infty$. Therefore by a result ([7], p. 286) due to Loève (or Theorem 2), (20) holds. Conversely, we will prove that $E(z^+ \log^+ z^+) < \infty$ and similarly for the negative part. Let $\mathcal{B}_n$ be the Borel field generated by the set $[z_n \geqq n]$. Then

$$E(z_n \mid \mathcal{B}_n) = I[z_n \geqq n] \textstyle\int_{[z_n \geqq n]} z_n/P[z_n \geqq n] + I[z_n < n] \int_{[z_n<n]} z_n/P[z_n < n]. \tag{22}$$

Since $\sum_1^{\infty} P[z_n \geqq n] = \sum_1^{\infty} P[z \geqq n] < \infty$, (20) implies that, by the Borel Cantelli lemma,

$$\textstyle\sum_1^{\infty} n^{-1} \int_{[z_n<n]} z_n/P[z_n < n] = -\sum_1^{\infty} n^{-1} \int_{[z_n \geqq n]} z_n/P[z_n < n] \tag{23}$$

converges a.e. Since $\lim_{n\to\infty} P[z_n < n] = 1$, $\sum_1^{\infty} \int_{[z_n \geqq n]} z_n/n = \sum_1^{\infty} \int_{[z \geqq n]} z/n$ converges. Now

$$\begin{aligned}\textstyle\sum_{n=1}^{\infty} \int_{[z \geqq n]} z/n &\geqq \textstyle\sum_{n=1}^{\infty} n^{-1} \sum_{j=n}^{\infty} jP[j \leqq z < j+1] \\ &\geqq \tfrac{1}{4} \textstyle\sum_{j=1}^{\infty} (j \log j)P[j \leqq z < j+1].\end{aligned}$$

Hence $E(z^+ \log^+ z^+) < \infty$ and the proof is completed.

It is interesting to compare Theorem 4 with a result [1] due to Burkholder, which states that under the assumption of Theorem 4, $E(|z| \log^+ |z|) < \infty$ if, and only if, for every sequence $\mathcal{B}_n$ of Borel fields such that $\mathcal{B}_n \subset \mathcal{A}_n$, we have

$$P[\lim n^{-1} \textstyle\sum_1^{n} E(z_m \mid \mathcal{B}_m) = 0] = 1, \tag{24}$$

provided that z has a continuous distribution.

3. Law of large numbers. We now turn to the law of large numbers.

THEOREM 5. *Let* $(y_n, \mathcal{F}_{n-1}, n \geqq 2)$ *be strictly positive stochastic sequence such that*

$$E(x_n y_n^{-1}) < \infty. \tag{25}$$

(a) *If* $1 \leqq p \leqq 2$, *then*

$$\lim s_n/y_n = 0 \tag{26}$$

a.e. *where*

$$\sum_2^\infty E(|x_n|^p \mid \mathfrak{F}_{n-1})y_n^{-p} < \infty, \qquad y_n \uparrow \infty. \tag{27}$$

(b) *If* $p > 2$, *then* (26) *holds* a.e. *where*

$$\sum_2^\infty E(|x_n|^p \mid \mathfrak{F}_{n-1})y_n^{-1-(p/2)} < \infty, \qquad y_n \uparrow \infty, \sum y_n^{-1} < \infty. \tag{28}$$

PROOF. Put $v_n = x_n/y_n$ and $u_n = v_2 + \cdots + v_n$. Then $(u_n, \mathfrak{F}_n, n \geqq 2)$ is a martingale. From Theorem 2, u_n converges a.e. where (27) holds if $1 \leqq p \leqq 2$. Now let $z_n = y_n^{-1}$. Then (14) is satisfied wherever (28) holds. By Theorem 3, u_n converges a.e. where (28) holds. From the Kronecker lemma ([7], p. 238), (26) holds a.e. where (27) and (28) holds.

COROLLARY 7. *Let* $1 \leqq p \leqq 2$ *and* $a_n = \sum_2^n E(|x_n|^p \mid \mathfrak{F}_{n-1}) < \infty$ a.e. *for* $n \geqq 2$. *Let* $f(t) \geqq 1$ *be a non-decreasing finite function on* $(0, \infty)$ *such that*

$$\int_0^\infty |f(t)|^{-p}\, dt < \infty. \tag{29}$$

Then

$$\lim s_n/f(a_n) = 0 \tag{30}$$

a.e. *where*

$$\lim a_n = \infty. \tag{31}$$

PROOF. Put $a_1 = 0$ and $y_n = f(a_n)$. Then y_n is $\mathfrak{F}_{n-1}$-measurable and

$$\sum_2^\infty E(|x_n|^p \mid \mathfrak{F}_{n-1})y_n^{-p} = \sum_2^\infty (a_n - a_{n-1})[f(a_n)]^{-p} \leqq \sum_2^\infty \int_{a_{n-1}}^{a_n} [f(t)]^{-p}\, dt = \int_0^\infty [f(t)]^{-p}\, dt < \infty.$$

By Theorem 5(a), we have that (30) holds a.e. where (31) is valid.

When $p = 2$, Corollary 7 reduces to a result of Lévy [5]. In ([8], p. 141) Neveu has derived Lévy's result from the law of large number for non-negative submartingales.

THEOREM 6. *Let* $(y_n, \mathfrak{F}_{n-1}, n \geqq 2)$ *be a strictly positive stochastic sequence such that* $y_n \uparrow \infty$ a.e. *Then* $\lim s_n/y_n = 0$ a.e. *where*

$$\sum_2^\infty y_n^{-2}E(x_n^2 I[|x_n| < y_n] \mid \mathfrak{F}_{n-1}) < \infty, \tag{32}$$

$$\lim y_n^{-1} \sum_2^n E(x_m I[|x_m| \geqq y_m] \mid \mathfrak{F}_{m-1}) = 0, \tag{33}$$

$$\limsup |x_n/y_n| < 1. \tag{34}$$

PROOF. Put $t_n = \sum_2^n y_m^{-1}\{x_m I[|x_m| < y_m] - E(x_m I[|x_m| < y_m] \mid \mathfrak{F}_{m-1})\}$. Then $(t_n, \mathfrak{F}_n, n \geqq 1)$ is a martingale, and

$$\sum_2^\infty E|(t_n - t_{n-1})^2 \mid \mathfrak{F}_{n-1}| \leqq \sum_2^\infty y_m^{-2}E(x_m^2 I[|x_m| < y_m] \mid \mathfrak{F}_{m-1}) < \infty$$

a.e. where (32) holds. By Corollary 5, t_n converges and then

$$\lim y_n^{-1} \sum_{m=2}^n \{x_m I[|x_m| < y_m] - E(x_m I[|x_m| < y_m] \mid \mathfrak{F}_{m-1})\} = 0$$

a.e. where (32) holds. Hence $\lim y_n^{-1} \sum_{m=2}^{n} x_m I[|x_m| < y_m] = 0$ a.e. where (32) and (33) hold. Therefore $\lim s_n/y_n = 0$ a.e. where (32)–(34) hold.

When $x_1, x_2, \cdots$ are independent, identically distributed with $E(x_1) = 0$ and $y_n = n$, Theorem 6 reduces to the usual proof ([6], p. 239) of Kolmogorov's strong law of large numbers.

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