THE BEHAVIOR OF LIKELIHOOD RATIOS OF STOCHASTIC PROCESSES RELATED BY GROUPS OF TRANSFORMATIONS

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Introduction. Let x_α be a 1-parameter family of stochastic processes and P_α the associated probability measures on the space of sample functions. We assume that the x_α are gotten from x_0 by the application of a group T_α of transformations, i.e., that T_α is a group of automorphisms on an algebra F, of bounded measurable functions dense in $L_1(P_0)$ and that $\int T_\alpha f\, dP_0 = \int f\, dP_\alpha$ for all f in F and all α.

In Section 2 we classify these problems as being conservative, dissipative, or mixed in analogy with terminology of ergodic theory. It turns out that many problems of interest are dissipative. Section 3 contains several such examples. Section 4 gives results on the spectrum of the associated isometries of $L_s(P_0)$ and on the asymptotic behavior of $dP_\alpha(x)/dP_0$ in the dissipative case.

2. The conservative and dissipative sets. Throughout this paper we will assume that the P_α are mutually absolutely continuous, that the T_α preserve bounds and either

(1) $T_\alpha f(x)$ has a continuous derivative $D(T_\alpha f)(x)$ in α which is bounded uniformly in α and x for every f in F and every x, or

(2) $T_\alpha f$ has an L_1-continuous L_1-derivative $DT_\alpha f$ for every f in F and $\|DT_\alpha f\| = O(e^{K|\alpha|})$ for some K independent of f.

We shall write P for P_0.

It has been shown [for condition (1) see [4] (Theorem 1, p. 272) and for condition (2) see [5] (Theorem 3.3)] that the above conditions imply that T_α can be extended to a group of automorphisms of all measurable functions and that the maps of $L_1(P)$, defined by

$$V_\alpha f = (dP_\alpha/dP),\, T_{-\alpha} f$$

form a strongly continuous 1-parameter group of isometries.

Thus $dP_\alpha/dP = V_\alpha(1)$ is L_1 continuous and it follows that we may regard dP_α/dP as a measurable stochastic process. By Fubini's theorem then $\int_{-T}^{T} [dP_\alpha(x)/dP]\, d\alpha$ exists and is finite for every finite T for almost all x. Set

$$q(x) = \int_{-\infty}^{\infty} [dP_\alpha(x)/dP]\, d\alpha.$$

We define the *conservative* set C to consist of those x with $q(x) = \infty$ and the *dissipative* set D to consist of those x with $q(x) < \infty$.

LEMMA 2.1. *The sets C and D are invariant under the T_α, to within sets of meas-*

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ure 0. *We have*

$$T_\alpha q(x) = [dP(x)/dP_{-\alpha}]q(x).$$

PROOF. Because of the L_1-continuity of dP_α/dP, the integral $\int_{-N}^{N} (dP_\alpha/dP)\, d\alpha$ can be approximated in $L_1(dP \otimes d\alpha)$ by Riemann sums and the formula for $T_\alpha q$ is established by using the relation $T_\alpha(dP_\beta/dP) = dP_{\beta-\alpha}/dP_{-\alpha}$ and then using standard approximation arguments. The invariance of C and D follows from this equation and the fact that $dP_\gamma/dP_\delta > 0$ almost everywhere for all γ and δ.

We will say that the system x_α is *conservative* if $P(C) = 1$, *dissipative* if $P(D) = 1$ and *mixed* otherwise. Any periodic system is obviously conservative. Another simple example of a conservative system is afforded by $x_\alpha(t) = x(t + \alpha)$, where x is a stationary process since then $dP_\alpha/dP = 1$ for all α. Let $x_\alpha(t) = x(t + \alpha)$, $-\infty < t < \infty$ be stochastic processes with associated measures P_α satisfying the conditions at the beginning of this section and with

$$\int |x(t)|\, dt < \infty$$

for almost all x. As will be seen in the next section, this implies that the system x_α is dissipative. Let Q be a stationary measure on the space of sample functions on the line. For any $0 < a < 1$ the measure $R = aP + (1 - a)Q$ has $dR_\alpha(x)/dR = dP_\alpha(x)/dP$ for x in L_1 and $R(L_1) = aP(L_1) = a$. Also $dR_\alpha(x)/dR = 1$ on a set C with $R(C) = (1 - a)Q(C) = 1 - a$. Hence, the system associated with R is mixed.

3. Examples of the dissipative case. Suppose we take the stochastic process x_0 to be a single random variable t (i.e., we take the parameter set to consist of a single point) and set $x_\alpha = t + \alpha$. If t is distributed according to the density $p(t)$ then $dP_\alpha(t)/dP = p(t - \alpha)/p(t)$. Thus, this case is dissipative with $q(t) = [p(t)]^{-1}$ and the measure $dQ = q\, dP$ is invariant under T_α.

The *mean value* case is generated by the transformations $(T_\alpha x)(t) = x_\alpha(t) = x(t) + \alpha f(t)$.

THEOREM 3.1. *The mean value case is dissipative.*

PROOF. Let t be a point where $f(t) \neq 0$—we will assume $f(t) > 0$. Then, since the measures P_α are mutually absolutely continuous, $x(t) \neq 0$ with probability 1. Let $\psi_{a,b}$ be the characteristic function of the set where $a \leqq x(t) < b$. Then

$$\begin{aligned}\int \psi_{a,b}(x)q(x)P(dx) &= \int_{-\infty}^{\infty} d\alpha \int \psi_{a,b}(x)P_\alpha(dx)\\ &= \int_{-\infty}^{\infty} d\alpha \int \psi_{a-\alpha f(t),b-\alpha f(t)}(x)P(dx)\\ &= \int P(dx) \int_{-\infty}^{\infty} \psi_{a-\alpha f(t),b-\alpha f(t)}(x)\, d\alpha\\ &= \int [(b - a)/f(t)]P(dx) = (b - a)/f(t) < \infty.\end{aligned}$$

Hence, taking $a = -N$ and $b = N$, q is finite almost everywhere on the set where $-N \leqq x(t) < N$ and, letting $N \to \infty$, q is therefore finite almost everywhere.

If we take x_0 in the above theorem to be a Gaussian process on a finite interval

T with mean 0 and correlation function $R(s, t)$ then, writing R for the integral operator associated with the correlation function, f must be in the range of $R^{\frac{1}{2}}$ for the P_α to be absolutely continuous [6]. Let (f_n) be a sequence from the range of R which approximates f. It can be shown that the random variables θ_n,

$$\theta_n(x) = \int_T x(t)(R^{-\frac{1}{2}}f_n)(t)\, dt,$$

converge in mean to a random variable $\theta(x)$ and that

$$dP_\alpha(x)/dP = \exp\,[\alpha\theta(x) - \tfrac{1}{2}\alpha^2\|R^{-\frac{1}{2}}f\|^2].$$

Hence for this case, $q(x) = (2\pi)^{\frac{1}{2}}\|R^{-\frac{1}{2}}f\|^{-1}\exp\,(\theta^2(x)/2\|R^{-\frac{1}{2}}f\|^2)$.

The transformation, $(T_\alpha x)(t) = x_\alpha(t) = e^{\alpha a(t)}x(t)$, is called a *Doppler shift* [3]. It is usually applied to complex processes. If $a(t) = i\lambda$, then T_α is periodic and we are in the conservative case.

THEOREM 3.2. *If $x_\alpha(t) = e^{\alpha a(t)}x(t)$, where x is a complex process, and there is a point t at which $\Re(a(t)) \neq 0$ and $|x(t)| \neq 0$ with probability one, then the system is dissipative.*

PROOF. Assume that $\Re(a(t)) = \gamma > 0$. Writing $\psi_{a,b}$ for the characteristic function of the set where $a \leqq |x(t)| < b$ and proceeding as in the previous proof we get $\int q(x)\psi_{a,b}(x)P(dx) = (\log b - \log a)/\gamma < \infty$. It follows as before that q is finite almost everywhere.

We have already seen that a *translation system*

$$(T_\alpha x)(t) = x_\alpha(t) = x(t + \alpha)$$

is conservative as x is stationary. The following theorem shows, on taking $\theta(r)$ very small for small r, that most translation systems in which $\lim_{t\to\pm\infty} x(t) = 0$ are dissipative. Taking $\theta(r)$ very small for large r shows that most translation systems for which $\lim_{t\to\pm\infty}|x(t)| = \infty$ are dissipative.

THEOREM 3.3. *Let x_α be a translation system and t a point where $x(t) \neq 0$ with probability* 1. *If there is some measurable function θ of a real variable satisfying*

(i) $\theta(r) > 0$ if $r \neq 0$ *and*

(ii) $G(x) = \int_{-\infty}^{\infty}\theta(x(s))\,ds\ \varepsilon\ L_1(P)$

then the system is dissipative.

PROOF. We have $\theta(x(t)) > 0$ almost everywhere and

$$\int q(x)\theta(x(t))P(dx) = \int P(dx)\int_{-\infty}^{\infty}\theta(x(t + \alpha))\,d\alpha = \int G(x)P(dx) < \infty,$$

so q is finite almost everywhere.

The list of examples in this section is intended to be illustrative not exhaustive. It seems reasonable to conjecture that many parameter estimation problems lead to the dissipative case though it might take more refined methods than those used above to prove this in some cases.

4. Special results for the dissipative case. Throughout this section we assume that, in addition to the assumptions listed at the beginning of this paper,

$$q(x) = \int_{-\infty}^{\infty}(dP_\alpha(x)/dP)\,d\alpha < \infty$$

almost everywhere, i.e., that we are in the dissipative case. Let Q be the σ-finite measure defined by setting $dQ = q\,dP$. A trivial application of Fubini's theorem shows that Q is invariant under T_α. Hence, T_α gives rise to a group of isometries of $L_s(Q)$ for every s. We will use the symbol T_α for these groups also.

For $1 \leqq s < \infty$ we define

$$V_\alpha^{(s)}: L_s(P) \to L_s(P); \qquad V_\alpha^{(s)}(f) = (dP_\alpha/dP)^{1/s} T_{-\alpha} f,$$

$$\xi_s : L_s(P) \to L_s(Q); \qquad \xi_s(f) = q^{-1/s} f.$$

It is easily verified that $V_\alpha^{(s)}$ is a strongly continuous 1-parameter group of isometries, that ξ_s is an isometry of $L_s(P)$ onto $L_s(Q)$, and that $V_\alpha^{(s)} = \xi_s^{-1} \circ T_{-\alpha} \circ \xi_s$.

THEOREM 4.1. *In the dissipative case the spectrum of $V_\alpha^{(2)}$ is all λ of absolute value 1 for all $\alpha \neq 0$.*

PROOF. Since $V_\alpha^{(2)} = \xi_2^{-1} \circ T_{-\alpha} \circ \xi_2$, the spectrum of V_α is the same as the spectrum of $T_{-\alpha}$ considered as a unitary operator on $L_2(Q)$. By a theorem of A. Ionescu Tulcea ([2], Corollary 2, p. 287) it will be sufficient to prove that $T_{-\alpha}$ is not periodic. But $(T_{-\alpha})^n = T_{-n\alpha} = I$ would imply that dP_β/dP had period $n\alpha$ and hence, that the system was conservative.

We will need the following continuous version of the Chacon-Ornstein ergodic theorem.

LEMMA 4.1. *Let σ be a σ-finite measure on a measure space (X, s). If $(V_\alpha\,; \alpha \geqq 0)$ is a strongly continuous semigroup of operators on $L_1(\sigma)$ satisfying*

(i) $\|V_\alpha\| \leqq 1$,

(ii) *if $f \geqq 0$ almost everywhere so is $V_\alpha f$,*

then for any f in $L_1(\sigma)$ and non-negative p in $L_1(\sigma)$, $\lim_{T\to\infty} (\int_0^T V_\alpha(f)(x)\,d\alpha / \int_0^T V_\alpha(p)(x)\,d\alpha)$ exists and is finite for almost every x for which $\int_0^\infty V_\alpha(p)(x)\,d\alpha > 0$.

PROOF. We may assume that f is non-negative. Now $\bar{f}$ and $\bar{p}$,

$$\bar{f} = \int_0^1 V_\alpha(f)\,d\alpha, \qquad \bar{p} = \int_0^1 V_\alpha(p)\,d\alpha,$$

are both in $L_1(\sigma)$ so by the Chacon-Ornstein theorem

$$\int_0^n V_\alpha(f)\,d\alpha / \int_0^n V_\alpha(p)\,d\alpha = \sum_{i=0}^{n-1} (V_1)^k \bar{f} / \sum_{i=0}^{n-1} (V_1)^k \bar{p}$$

converges to a finite limit almost everywhere on the set where $\sum_{i=0}^{\infty} (V_i)^k \bar{p} = \int_0^\infty V_\alpha(p)\,d\alpha > 0$. Writing $[T]$ for the greatest integer in T, we have, by Lemma 4 of [1], for any non-negative g in $L_1(\sigma)$,

$$\int_{[T]}^T V_\alpha(g)\,d\alpha / \sum_{i=0}^{[T]-1} (V_1)^k \bar{g} \leqq (V_1)^n \bar{g} / \sum_{i=0}^{[T]-1} (V_1)^k \bar{g} = \epsilon(g, T, x) \to 0$$

for almost every x as $T \to \infty$. Hence,

$$\lim_{T\to\infty} \frac{\int_0^T V_\alpha(f)(x)\,d\alpha}{\int_0^T V_\alpha(p)(x)\,d\alpha} = \lim_{T\to\infty} \frac{(1 + \epsilon(f, T, x)) \sum_{i=0}^{n-1} (V_1)^i(f)(x)}{(1 + \epsilon(p, T, x)) \sum_{i=0}^{n-1} (V_1)^i(p)(x)}$$

$$= \lim_{T\to\infty} \sum_{i=0}^{n-1} (V_1)^i(f)(x) \Big/ \sum_{i=0}^{n-1} (V_1)^i(p)(x)$$

almost everywhere where the latter limit exists.

LEMMA 4.2. *In the dissipative case $V_\alpha(f)(x)$ is integrable in α for almost every x if f is in $L_1(P)$.*

PROOF. Assuming f is non-negative,

$$\lim_{T\to\infty} \int_0^T V_\alpha(f)(x)\, d\alpha$$

$$= \int_0^\infty (dP_\alpha(x)/dP)\, d\alpha \lim_{T\to\infty} \int_0^T V_\alpha(f)(x)\, d\alpha / \int_0^T V_\alpha(1)(x)\, d\alpha$$

is finite almost everywhere. Replacing V_1 by V_{-1} shows that $\int_{-\infty}^0 V_\alpha(f)(x)\, d\alpha$ is also finite almost everywhere and completes the proof.

THEOREM 4.2. *$V_\alpha^{(s)}$ has no non-trivial eigenvectors for $1 \leqq s < \infty$ and $\alpha \neq 0$ in the dissipative case.*

PROOF. We will prove the theorem first for $s = 1$ and assume that $\alpha > 0$. If $V_\alpha(f) = \theta f$ then $|\theta| = 1$ since V_α is an isometry. Let χ be a function of absolute value 1 such that $\chi f = |f|$. Then

$$V_\alpha(|f|) = V_\alpha(\chi f) = (dP_\alpha/dP)T_{-\alpha}(\chi f) = (T_{-\alpha}\chi)V_\alpha(f)$$

$$= (T_{-\alpha}\chi)\theta f = |f|$$

since $|(T_{-\alpha}\chi)\theta| = 1$ almost everywhere and $V_\alpha(|f|) \geqq 0$. Thus

$$\int_0^{n\alpha} V_\beta(|f|)\, d\beta = n \int_0^\alpha V_\beta(|f|)\, d\beta,$$

so $\int_0^\alpha V_\beta(|f|)\, d\beta$ and hence, $|f|$ is 0 almost everywhere by Lemma 4.2. Finally, if $s > 1$ and $V_\alpha^{(s)}(f) = \theta f$ then

$$V_\alpha(f^s) = [V_\alpha^{(s)}(f)]^s = (\theta f)^s = \theta^s f^s,$$

so $f = f^s = 0$.

Our final theorem concerns the asymptotic values of $dP_\alpha(x)/dP$. It is desirable in maximum likelihood testing for these limits to be 0 since otherwise maxima may occur for arbitrarily large α. Considering the example at the beginning of Section 3 where $dP_\alpha(t)/dP = p(t - \alpha)/p(t)$ it is clear that some further assumption is required to assure this.

THEOREM 4.3. *In the dissipative case, if there exists a φ in $L_1(P)$ satisfying*

$$\int \varphi f\, dP = (\partial/\partial\alpha) \int T_\alpha f\, dP\, |_{\alpha=0}$$

for all f in F then $dP_\alpha(x)/dP$ is a differentiable function of α tending to 0 as $\alpha \to \pm\infty$ for almost every x.

PROOF. It is known (see [4], Theorem 1, p. 272 for condition (1) and [5], Theorem 3.3 for condition (2)) that we may take

$$dP_\alpha(x)/dP = 1 + \int_0^\alpha V_\beta(\varphi)(x)\, d\beta$$

in this case. By Lemma 4.2 $\lim_{\alpha\to\infty} dP_\alpha(x)/dP$ exists for almost every x and this limit must be 0 since $dP_\alpha(x)/dP$ is almost always integrable in α.

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