ON THE MEAN NUMBER OF CURVE CROSSINGS BY NON-STATIONARY NORMAL PROCESSES[1]

By M. R. Leadbetter and J. D. Cryer

Research Triangle Institute

Summary. In this paper we consider the mean number of crossings of an arbitrary curve, in a given time T, by a non-stationary normal process. A formula (2) is obtained for this and sufficient conditions for its validity given. These conditions concern the behaviour of the covariance function of the normal process. Incidental results include sufficient conditions for continuity of a normal (non-stationary) process $x(t)$ and also for $x(t)$ to satisfy a Hölder condition.

1. Introduction. The problem of obtaining the mean number of crossings of a *fixed* level u, by a *stationary* normal process has received a great deal of attention in the literature. The most recent work we are aware of in this connection is that due to Bulinskaya (1961) who derives the well known formula

$$\mathcal{E}\{N(T)\} = T\pi^{-1}\{-r''(0)/r(0)\}^{\frac{1}{2}}e^{-u^2/(2r(0))} \tag{1}$$

under conditions which are very close to the necessary ones. Here $N(T)$ is the number of crossings of the level u in $(0, T)$ by the stationary normal process $\{y(t)\}$, with covariance function $r(t)$.

Cramér (1963) considered the integral $x(t) = \int_0^t y(s)\,ds$ of a normal stationary process $\{y(t)\}$ and obtained the formula corresponding to (1) for the mean number of crossings of a fixed level u, during time T, in this case. The process $\{x(t)\}$ is, of course, normal but, in general, non-stationary.

Finally, in this connection, this problem has been considered (Leadbetter, 1965) in the case where the fixed level u is replaced by a curve $u(t)$ and where the normal process $\{x(t)\}$ is either stationary or is the integral of a normal stationary process. The methods of Bulinskaya were employed to give quite weak sufficient conditions under which the appropriate modification of (1) holds in these two cases.

When one considers a general non-stationary normal process $\{x(t)\}$ it is immaterial whether one looks at crossings of an arbitrary curve $u(t)$ or merely crossings of the axis, i.e. $u(t) \equiv 0$. For the former case may be reduced to the latter by simply subtracting the quantity $u(t)$ and considering the normal process $x(t) - u(t)$. We shall assume throughout that this has been done and we may thus restrict our attention to the case of axis crossings by a normal process $\{x(t)\}$.

In Section 2 we obtain the formula (Theorem 1) corresponding to (1), for the mean number of axis-crossings (and hence, as noted, the mean number of curve

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crossings) by a non-stationary normal process. This theorem is stated for a normal process $\{x(t)\}$ having, almost surely, a continuous sample derivative. For stationary normal processes, and their sample integrals, very weak conditions—due to Hunt (1951), Belaev (1961)—are sufficient to ensure that the process has, a.s. a continuous sample derivative. In Section 3 we give sufficient conditions for this to hold in the general (non-stationary) case. These sufficient conditions are very little stronger than those of Hunt-Belaev, and it would be interesting to know whether they can be weakened to correspond to the Hunt-Belaev ones. Sufficient conditions for the process sample functions to satisfy a Hölder condition, are also given in Section 4.

2. Mean number of curve crossings. Let $\{x(t): 0 \leqq t \leqq T\}$ be a separable, normal stochastic process with $\mathcal{E}\{x(t)\} = m(t)$, cov $\{x(t), x(t')\} = \Gamma(t, t')$. In considering crossings of a (continuously differentiable) curve $u(t)$, there is, as already explained, no loss of generality in taking $u(t) \equiv 0$, and we do this. We have then the following result.

THEOREM 1. *Suppose that $m(t)$ has a continuous derivative $m'(t)$ for $0 \leqq t \leqq T$, that Γ has a mixed second partial derivative, which is continuous at all diagonal points (t, t), $0 \leqq t \leqq T$, and that $x(t)$ has,* a.s., *a continuous sample derivative $x'(t)$ in $0 \leqq t \leqq T$. (Theorem 3 gives sufficient conditions for this latter property.) Assume also that the joint distribution of $x(t)$ and $x'(t)$ is non-degenerate for any t in $0 \leqq t \leqq T$. Then*

$$\mathcal{E}\{N(T)\} = \int_0^T \gamma\sigma^{-1}(1 - \rho^2)^{\frac{1}{2}}\phi(m/\sigma)\{2\phi(\eta) + \eta(2\Phi(\eta) - 1)\}\, dt \tag{2}$$

where

$$\sigma^2 = \sigma^2(t) = \text{var}\,\{x(t)\} = \Gamma(t, t)$$

$$\gamma^2 = \gamma^2(t) = \text{var}\,\{x'(t)\} = \partial^2\Gamma/\partial t\partial t'|_{t'=t}$$

$$\rho = \rho(t), \qquad \gamma\sigma\rho = \text{cov}\,(x(t), x'(t)) = \partial\Gamma/\partial t'|_{t'=t}$$

and $\eta = \eta(t) = (m' - \gamma\rho m/\sigma)/[\gamma(1 - \rho^2)^{\frac{1}{2}}]$, writing m for $m(t)$, m' for $m'(t)$. (ϕ and Φ are the standard normal density and distribution function respectively.)

The proof of this theorem will be by means of a series of lemmas. Some of these are given by Leadbetter (1965) but are repeated here for completeness. However, we first develop some notation.

There is no loss of generality in taking $T = 1$, and we do so. It is convenient to use the method of Bulinskaya (1961) in approximating the $x(t)$-process by a sequence of processes consisting of straight line segments, as follows:

For each positive integer n, and each t in $0 \leqq t \leqq 1$ let $k_n = k_n(t)$ be the unique integer such that $k_n/2^n \leqq t < (k_n + 1)/2^n$, $(0 \leqq k_n \leqq 2^n)$. Write $y_n(t) = x(k_n/2^n) + 2^n[x((k_n + 1)/2^n) - x(k_n/2^n)](t - k_n/2^n)$. That is $\{y_n(t)\}$ is a new process coinciding with $x(t)$ at points $t = k/2^n$, and consisting of straight line segments between such points. Write also N_x for $N(1)$ and N_{y_n} for the number of axis crossings by the y_n-process in $0 \leqq t \leqq 1$. We may apply Theorem 1 of Bulinskaya (1961) since continuity of $m(t)$, $\Gamma(t, t)$, together with the assumption $\Gamma(t, t) > 0$, imply that the univariate probability density for $x(t)$ is bounded in $0 \leqq t \leqq 1$. Hence it follows that N_x is finite with probability one and the event that $x(t)$ be tangential to the axis somewhere, has probability zero. It is thus clear that N_{y_n} increases to the limit N_x, with probability one, as $n \to \infty$. Hence, by monotone convergence we have

LEMMA 1.

$$\mathcal{E}\{N_{y_n}\} \to \mathcal{E}\{N_x\} \textit{ as } n \to \infty.$$

To evaluate $\mathcal{E}\{N_{y_n}\}$ we use a sequence of functions "approaching a Dirac-δ." Specifically we shall here call a sequence $\{\delta_v(x)\}$ of non-negative integrable functions a *δ-function sequence* if $\int_{-\infty}^{\infty} \delta_v(x)\, dx = 1$ for all $v = 1, 2, \cdots$ and $\int_{-\lambda}^{\lambda} \delta_v(x) \cdot dx \to 1$ as $v \to \infty$ for any fixed $\lambda > 0$. Then we have

LEMMA 2. *With probability one,*

$$N_{y_n} = \lim_{v\to\infty} \int_0^1 \delta_v\{y_n(t)\}\,|y_n'(t)|\,dt$$

and

$$\int_0^1 \delta_v\{y_n(t)\}\,|y_n'(t)|\, dt \leqq 2^n.$$

PROOF. Write $\alpha_k = k/2^n$ and $y_n(t) = A_k + B_k t$ for $\alpha_k \leqq t \leqq \alpha_{k+1}$. Then

$$\int_0^1 \delta_v(y_n(t))\,|y_n'(t)|\, dt = \sum_{k=0}^{2^n-1} \int_{\alpha_k}^{\alpha_{k+1}} \delta_v(A_k + B_k t)\,|B_k|\, dt = \sum_{k=0}^{2^n-1} \left| \int_{y_n(\alpha_k)}^{y_n(\alpha_{k+1})} \delta_v(x)\, dx \right|. \tag{3}$$

With probability one $y_n(\alpha_k)$ is not zero for any $k = 0, 1 \cdots 2^n$. From the assumed δ-function properties it follows that if $y_n(\alpha_k)$ and $y_n(\alpha_{k+1})$ have the same sign, the corresponding integral tends to zero. Otherwise, this integral converges to ± 1. Thus, if the interval (α_k, α_{k+1}) contains a zero of $y_n(t)$, the corresponding term in the sum tends to one, and otherwise it tends to zero. Hence, the first part of the lemma follows. The second part follows from (3) since each term in the sum is dominated by $\int_{-\infty}^{\infty} \delta_v(x)\, dx = 1$.

By virtue of this lemma it follows by dominated convergence (and Fubini's theorem for positive functions) that

$$\mathcal{E}\{N_{y_n}\} = \lim_{v\to\infty} \int_0^1 \mathcal{E}\{\delta_v(y_n(t))\,|y_n'(t)|\}\, dt = \lim_{v\to\infty} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |w|\,\delta_v(v)\,p_n(v, w)\, dv\, dw\, dt, \tag{4}$$

where $p_n(v, w)$ is the bivariate normal density function for $(y_n(t), y_n'(t))$, having the form

$$p_n(v, w) = (2\pi D^{\frac{1}{2}})^{-1} \exp\left[-\{C(v-\alpha)^2 - 2B(v-\alpha)(w-\beta) + A(w-\beta)^2\}/(2D)\right] \tag{5}$$

in which

$$\alpha = \alpha_n(t) = \mathcal{E}\{y_n(t)\}, \qquad \beta = \beta_n(t) = \mathcal{E}\{y_n'(t)\}$$

$$A = A_n(t) = \text{var}\,\{y_n(t)\}, \qquad C = C_n(t) = \text{var}\,\{y_n'(t)\}$$
$$B = B_n(t) = \text{cov}\,\{y_n(t), y_n'(t)\} \qquad D = D_n(t) = AC - B^2.$$

We note that, for any probability density function $h(t)$, we may obtain a δ-function sequence $\{\delta_v\}$ defined by $\delta_v(t) = vh(vt)$. In particular if we use the normal density $h(t) = (2\pi)^{-\frac{1}{2}}e^{-t^2/2}$ we obtain from (4),

$$(6) \qquad \mathcal{E}\{N_{y_n}\} = \lim_{v\to\infty} (2\pi)^{-\frac{1}{2}} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-v^2/2}|w|p_n(v/v, w)\, dv\, dw\, dt.$$

In order to simplify this expression and to obtain its limit as $n \to \infty$ we require the following result.

LEMMA 3.

We have the following uniform limits in $0 \leqq t \leqq 1$:
(i) $\alpha_n(t) \to m(t)$ (ii) $\beta_n(t) \to m'(t)$ (iii) $A_n(t) \to \Gamma(t, t)$
(iv) $B_n(t) \to \Gamma_{01}(t, t)$ (v) $C_n(t) \to \Gamma_{11}(t, t)$
(vi) $D_n(t) \to \Gamma(t, t)\Gamma_{11}(t, t) - \Gamma_{01}^2(t, t)$ *where* $\Gamma_{01} = \partial\Gamma(t, t')/\partial t'$, *etc.*

PROOF. Since $\Gamma_{11}(t, t')$ exists for $0 \leqq t, t' \leqq 1$ and is continuous where $t = t'$ it follows (see, for example, Loève (1963), Section 34.2B) that $\Gamma_{11}(t, t')$ is continuous for $0 \leqq t, t' \leqq 1$. Thus $\Gamma_{11}(t, t')$, $\Gamma_{10}(t, t')$, and $\Gamma_{01}(t, t')$ are all uniformly continuous and bounded for $0 \leqq t, t' \leqq 1$. Using these facts and the definition of $y_n(t)$, the required limits are then found by straightforward (if somewhat tedious) applications of the mean value theorem.

PROOF OF THE THEOREM. The integrand in the expression (6) for $\mathcal{E}\{N_{y_n}\}$ is dominated by $(2\pi D^{\frac{1}{2}})^{-1}|w| \exp\{-\frac{1}{2}[v^2 + (w - \beta)^2/2C]\}$ and converges to $|w|e^{-v^2/2}p_n(0, w)$ as $v \to \infty$. But it is clear from the calculations of Lemma 3 that β and C are bounded functions of t, for any n. Further since the uniform limit (vi) of D is non-zero (by the assumed non-degeneracy) it follows that, at least for sufficiently large n, D is bounded away from zero in $0 \leqq t \leqq 1$. Hence, by dominated convergence

$$\mathcal{E}\{N_{y_n}\} = \int_0^1 \int_{-\infty}^{\infty} |w|p_n(0, w)\, dw\, dt.$$

Using Equation (5) we obtain, after some reduction,

$$\mathcal{E}\{N_{y_n}\} = (2/\pi)^{\frac{1}{2}} \int_0^1 (D^{\frac{1}{2}}/A)e^{-\alpha^2/(2A)}[\phi(\omega) + \omega(\Phi(\omega) - \tfrac{1}{2})]\, dt$$

in which $\omega = \omega_n(t) = (A/D)^{\frac{1}{2}}(\beta - B\alpha/A)$. Using the limits of Lemma 3 (from which it follows in particular that $\omega_n(t) \to \eta(t)$), and bounded convergence, the required result follows.

3. Continuity and differentiability. In proving Theorem 1, it was assumed that the process $\{x(t)\}$ has, a.s., a continuous sample derivative. For *stationary* normal processes, a very weak sufficient condition for this property is available from the work of Hunt (1951). Hunt's condition is expressed in terms of the spectrum of the process. An equivalent condition in terms of the covariance function has been given by Belaev (1961, Eqn. 45). This result states that for a normal stationary process $x(t)$ to have a.s., a continuous sample function it is sufficient that the covariance function $r(\tau)$ satisfy

$$(1 - r(\tau)) \leqq C/|\log|\tau||^a \tag{7}$$

for some $a > 1$, $C > 0$, and all sufficiently small τ. For $x(t)$ to have, a.s., a continuous sample derivative, it is sufficient that the second derivative $r''(\tau)$ exist and (7) hold with $1 - r(\tau)$ replaced by $-r''(0) + r''(\tau)$, ($-r''(\tau)$ is the covariance function of the quadratic mean derivative of $x(t)$).

The proof given by Belaev rests on Hunt's result which, in proof, makes essential use of the stationarity of the process. We shall now prove a result corresponding to that of Belaev (but not quite as strong), for non-stationary normal processes. Specifically it will be assumed that an equation generalizing (7) holds, but with the constant $a > 3$ rather than just $a > 1$. Nevertheless this is still a very weak assumption. The proof of this result will be a direct application of the theorem of "Kolmogorov-Slutsky type" in the form given by Loève (1963), (Sample continuity moduli theorem). From this result a sufficient condition for the normal, non-stationary process $x(t)$ to have, a.s., a continuous sample derivative (as required in Theorem 1) can be obtained.

THEOREM 2. *Let* $\{x(t): 0 \leqq t \leqq T\}$ *be a separable, normal process with* $\mathcal{E}\{x(t)\} = m(t)$, *continuous, and continuous covariance function* $\Gamma(t, t')$. *Write* $\Delta_h\Gamma(t, t') = \Gamma(t + h, t' + h) - \Gamma(t + h, t') - \Gamma(t, t' + h) + \Gamma(t, t')$ *and suppose that for* $0 \leqq t \leqq T$, $\Delta_h\Gamma(t, t) \leqq C|\log|h||^{-a}$ *for some* $C > 0$, $a > 3$, *and all sufficiently small* h. *Then* $x(t)$ *has*, a.s. *continuous sample functions.*

PROOF. There is no loss of generality in taking $m(t) \equiv 0$ since continuity of $m(t)$ implies that $x(t)$ has a continuous sample function if and only if $x(t) - m(t)$ does. Using the notation of Loève (1963, p. 517) we write

$$g(h) = |\log|h|/\log 2|^{-\beta}$$

where β is chosen so that $1 < \beta < (a - 1)/2$, $(a > 3)$. Now, writing $\mathcal{E}\{x(t + h) - x(t)\}^2 = \sigma_h^2 (= \Delta_h\Gamma(t, t))$, we have

$$\begin{aligned}
\Pr\{|x(t + h) - x(t)| \geqq g(h)\} &= 2\{1 - \Phi(g(h)/\sigma_h)\} \\
&\leqq 2\{1 - \Phi(g(h)\,|\log|h||^{a/2}/C^{\frac{1}{2}})\} \\
&\leqq \phi\{C^{-\frac{1}{2}}g(h)\,|\log|h||^{a/2}\}\{2C^{\frac{1}{2}}\,|\log|h||^{-a/2}/g(h)\} \\
&= q(h), \qquad \text{say}.
\end{aligned}$$

Now $g(2^{-n}) = n^{-\beta}$ and hence $\sum_{n=1}^{\infty} g(2^{-n}) < \infty$. Further

$$\begin{aligned}
q(2^{-n}) &= \text{const. } n^{\beta - a/2} \exp\{-(n^{-2\beta}(n \log 2)^a)/2C\} \\
&= \text{const. } n^{\beta - a/2}\rho^{n^{a - 2\beta}}, \qquad \text{where} \quad 0 < \rho < 1.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} 2^n q(2^{-n}) &\leqq \text{const. } \sum_{n=1}^{\infty} n^{\beta - a/2} 2^n \rho^{n^{a - 2\beta}} \\
&\leqq \text{const. } \sum_{n=1}^{\infty} \rho^n \qquad (\text{since } a - 2\beta > 1) \\
&< \infty.
\end{aligned}$$

Hence the conditions for the "Sample continuity moduli theorem" of Loève (loc. cit.) are satisfied and $x(t)$ has, a.s. continuous sample functions.

Using this result, we may now obtain sufficient conditions for $x(t)$ to have, a.s., a continuous sample derivative, as follows.

THEOREM 3. *Let $\{x(t): 0 \leqq t \leqq T\}$ be a separable, normal process with mean $\mathcal{E}\{x(t)\} = m(t)$ and covariance function $\Gamma(t, t')$. Let $m(t)$ have a continuous derivative $m'(t)$ in $0 \leqq t \leqq T$. Let Γ have a continuous mixed second derivative $\Gamma_{11}(t, t') = \partial^2\Gamma/\partial t\partial t'$ satisfying for some constants $C > 0$, $a > 3$, $0 \leqq t \leqq T$, and all sufficiently small h*

$$\Delta_h\Gamma_{11}(t, t) \leqq C\,|\log|h||^{-a}.$$

Then $x(t)$ has a.s., *a sample derivative $x'(t)$ which is continuous on $0 \leqq t \leqq T$.*

PROOF. For convenience we may again take $m(t) = 0$, since $x(t)$ possesses a continuous sample derivative if and only if $x(t) - m(t)$ does. Since Γ_{11} is continuous, $x(t)$ has a quadratic mean (q.m.) derivative $x'(t)$ with Γ_{11} as its covariance function. This follows, for example, from Section 34.2C of Loève (1963). Also from Section 35.2E of the same reference it is evident that we may take a separable, "a.e. Borel" version of $x'(t)$.

Write now $x(t, \omega)$, $x'(t, \omega)$ to exhibit explicitly dependence on the "sample point" ω. Then it follows from Theorem 2 above that $x(t, \omega)$ and $x'(t, \omega)$ are continuous functions of t for each ω outside a null ω-set N_1. Write $y(t, \omega)$ for the sample integral $\int_0^t x'(s, \omega)\, ds$, $\omega \not\in N_1$. Then it follows from the "Second order calculus theorem," (i), of Loève (loc. cit., Section 35.3C) that, for each t, $y(t, \omega) = x(t, \omega) - x(0, \omega)$, a.s. Hence if S is a countable dense subset of $[0, T]$, we can find a null ω-set N_2, (taken to include N_1), such that

$$y(t, \omega) = x(t, \omega) - x(0, \omega), \qquad t \in S, \quad \omega \not\in N_2 .$$

Continuity of both x and y then shows that this relation is true for all t in the interval $[0, T]$, and ω outside N_2. But since for each $\omega \not\in N_2$, the derivative of $\int_0^t x'(s, \omega)\, ds$ is the (continuous) function $x'(t, \omega)$ it follows that $y(t, \omega)$ has, for $\omega \not\in N_2$, the continuous sample derivative $x'(t, \omega)$. Hence finally $x(t) = y(t) + x(0)$ has, a.s., the continuous sample derivative $x'(t)$.

As remarked already, this sufficient condition may be used in Theorem 1 instead of the condition that $x(t)$ have a continuous sample derivative, which appears there. We thus have criteria for the validity of Theorem 1, based on the nature of the covariance function Γ.

4. A Hölder condition for normal sample functions. In this section we give sufficient conditions for the sample functions of a normal process to satisfy a Hölder condition. This has no direct bearing on the main part of the paper—namely Theorem 1, but is, of course, related to Theorem 2. Naturally the conditions required here are a little more restrictive than those of Theorem 2. The result we shall obtain generalizes one of Belaev (1961, Theorem 7) which was given for the stationary case. (In fact our result is also a little better than that of Belaev, even in the stationary case.)

THEOREM 4. *Let* $\{x(t): 0 \leqq t \leqq T\}$ *be a normal process,* $\mathcal{E}\{x(t)\} = m(t)$, $\operatorname{cov}\{x(t), x(t')\} = \Gamma(t, t')$. *Suppose that* Γ *satisfies the condition* $\Delta_h\Gamma(t, t) \leqq C|h|^{2a}/|\log|h||$, $a > 0$, $C > 0$, *all sufficiently small* h, *and* $0 \leqq t \leqq T$. *Suppose also that for some constant* B *there exists* $\delta > 0$ *such that for* $|h| < \delta$, $|m(t+h) - m(t)| < B|h|^a$ (i.e. *m satisfies a Hölder condition*). *Then, given* $\epsilon > 0$, *there exists an* a.s. *positive random variable* H_ϵ *such that, if* $|h| < H_\epsilon$,

$$|x(t + h) - x(t)| < [(2C)^{\frac{1}{2}} + B](1 + \epsilon)|h|^a.$$

PROOF. Write $x^*(t) = x(t) - m(t)$, $g(h) = C_1|h|^a$ (C_1 to be chosen later),

$$\sigma_h^2 = \Delta_h\Gamma(t, t) = \mathcal{E}\{x^*(t + h) - x^*(t)\}^2$$

Now

$$\begin{aligned}\Pr\{|x^*(t + h) - x^*(t)| > g(h)\} &= 2\{1 - \Phi(g(h)/\sigma_h)\}\\ &\leqq 2\{1 - \Phi(C_1\,|\log|h||^{\frac{1}{2}}/C^{\frac{1}{2}})\}\\ &\leqq \text{const.}\ |\log|h||^{-\frac{1}{2}} \exp\{-C_1^2\,|\log|h||/(2C)\}\\ &= q(h), \qquad \text{say.}\end{aligned}$$

Hence for any fixed integer j,

$$\begin{aligned}q(j2^{-n}) &= \text{const.}\ |n\log 2 - \log j|^{-\frac{1}{2}} \exp\{-C_1^2|n\log 2 - \log j|/(2C)\}\\ &\leqq \text{const.}\ n^{-\frac{1}{2}} \exp\{-C_1^2 n\log 2/(2C)\} \qquad \text{for sufficiently large } n.\end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} 2^n q(j2^{-n}) \leqq \text{const.} \sum_{n=1}^{\infty} n^{-\frac{1}{2}} 2^{n\{1 - C_1^2/(2C)\}}$$

which is convergent if $C_1^2 > 2C$. Further $\sum_{j=1}^{\infty} g(2^{-(n+j)})/g(2^{-n}) = \sum_{j=1}^{\infty} 2^{-ja}$, is bounded, independently of n. Finally $g(2^{-n})/g(j2^{-n}) = j^{-a}$ which is arbitrarily small for sufficiently large j. Hence, it follows from the "Sample continuity moduli theorem," (ii) of Loève (1963, p. 517) that, given $\epsilon > 0$, there exists an a.s. positive random variable H_ϵ such that, for $|h| < H_\epsilon$,

$$\begin{aligned}|x^*(t + h) - x^*(t)| &< (1 + \epsilon)g(h)\\ &= C_1(1 + \epsilon)|h|^a,\end{aligned}$$

provided $C_1^2 > 2C$. But, given $\epsilon > 0$, we may choose $C_1 > (2C)^{\frac{1}{2}}$, and $\epsilon_1 > 0$ such that $C_1(1 + \epsilon_1) < (2C)^{\frac{1}{2}}(1 + \epsilon)$. Using C_1 and ϵ_1 in the above statement, we see that, given $\epsilon > 0$, there exists an a.s. positive random variable H_ϵ such that, for $|h| < H_\epsilon$,

$$|x^*(t + h) - x^*(t)| < (2C)^{\frac{1}{2}}(1 + \epsilon)|h|^a.$$

The theorem then follows since

$$|x(t + h) - x(t)| \leqq |x^*(t + h) - x^*(t)| + |m(t + h) - m(t)|.$$

5. Further remarks. The purpose of the assumption in Theorem 1, that

$x(t)$ have a continuous sample derivative, was to ensure that there be zero probability of $x(t)$ becoming somewhere tangent to the axis (or curve). However if one restricts attention to the number $Z(T)$ of "genuine crossings" of the axis or curve—that is not counting tangencies as crossings—it is possible to dispense with this assumption. This was pointed out by Ylvisaker (1961) who derived Equation (1), with $Z(T)$ in place of $N(T)$, merely under the assumption that $r''(0)$ exists.

The same thing is true in the nonstationary case. In fact even if we omit the condition that $x(t)$ have a continuous sample derivative, it can be seen that $Z(T) = \lim_{n\to\infty} N_{y_n}$, a.s. It follows then that Theorem 1 is true for $Z(T)$ in place of $N(T)$ without the condition that $x(t)$ should possess, a.s., a continuous sample derivative. If $x(t)$ does possess, a.s., a continuous sample derivative, then of course $Z(T) = N(T)$, a.s. In fact Ylvisaker (private communication) has recently shown that the requirement of a continuous sample derivative is not a necessary condition for this equality in the stationary situation, and the indications are that the same is true in a wide variety of nonstationary situations. If this proves to be the case in general, it will be possible to further improve the conditions of Theorem 1 by weakening or omitting the requirement of a continuous sample derivative.

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