ON THE ASYMPTOTIC POWER OF THE ONE-SAMPLE KOLMOGOROV-SMIRNOV TESTS[1]

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1. Introduction. Let $X_1, X_2, \cdots, X_n$ be a random sample of n observations from some unknown distribution function F, and let

$$\begin{aligned} F_n(x) &= 0, & x &< X_{(1)}, \\ &= i/n, & X_{(i)} &\leqq x < X_{(i+1)}, \\ &= 1, & x &\geqq X_{(n)}, \end{aligned}$$

where $X_{(1)} \leqq X_{(2)} \leqq \cdots \leqq X_{(n)}$ are the ordered observations. If H is some completely specified continuous distribution function, we may reject the hypothesis that $F = H$ for large values of $K_n = \sup_{-\infty<x<\infty} n^{\frac{1}{2}} |F_n(x) - H(x)|$ or $K_n^+ = \sup_{-\infty<x<\infty} n^{\frac{1}{2}}[F_n(x) - H(x)]$; specifically, if $K_n \geqq d_n(\alpha)$ where $P\{K_n \geqq d_n \mid F = H\} = \alpha$ or if $K_n^+ \geqq d_n^+(\alpha)$ where $P\{K_n^+ \geqq d_n^+ \mid F = H\} = \alpha$. These probabilities do not depend on the true underlying distribution function, so long as it is continuous. The test based on K_n was first proposed in 1933 by Kolmogorov [9], and the related K_n^+ test was later suggested by Smirnov [12]; they are called respectively the two-sided and one-sided one-sample Kolmogorov-Smirnov tests of goodness of fit. For a fuller expository treatment we refer to the paper by Darling [5] which also includes an extensive bibliography.

The power of the K_n test when F is equal to some alternative continuous distribution function G is

$$P\{K_n \geqq d_n(\alpha) \mid F = G\} = P\{\sup_{-\infty<x<\infty} n^{\frac{1}{2}} |F_n(x) - H(x)| \geqq d_n(\alpha) \mid F = G\},$$

and if $\{G_n\}$ is some sequence of alternative distributions, we may define the asymptotic power against $\{G_n\}$ to be

$$\lim_{n\to\infty} P\{\sup_{-\infty<x<\infty} n^{\frac{1}{2}} |F_n(x) - H(x)| \geqq d_n(\alpha) \mid F = G_n\}$$

if this limit exists. Following Doob [7], we introduce the stochastic process $Z_n(t) = n^{\frac{1}{2}}(F_n[F^{-1}(t)] - t)$, $0 \leqq t \leqq 1$; then the asymptotic power may be rewritten in the form

$$\lim_{n\to\infty} P\{\sup_{0<t<1} |Z_n(t) - n^{\frac{1}{2}}(H[G_n^{-1}(t)] - t)| \geqq d_n(\alpha) \mid F = G_n\}.$$

We may omit the condition $F = G_n$ in this expression, since all probability state-

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ments about $Z_n(t)$ are entirely independent of F, so long as F is continuous; in what follows we shall for convenience take F to be the uniform distribution on the interval [0, 1]. Define also

$$C_n(t) = n^{\frac{1}{2}}(H[G_n^{-1}(t)] - t), \qquad n = 1, 2, \cdots . \tag{1.1}$$

Then the asymptotic power may be written as $P(\alpha, \{C_n\}) = \lim_{n\to\infty} P\{\sup_{0<t<1} \cdot |Z_n(t) - C_n(t)| \geqq d_n(\alpha)\}$. The notation here indicates that the power does not depend on H and $\{G_n\}$ separately, but only on the sequence $\{C_n\}$. In a similar manner we may define the asymptotic power of the K_n^+ test and write it as $P^+(\alpha, \{C_n\}) = \lim_{n\to\infty} P\{\sup_{0<t<1} [Z_n(t) - C_n(t)] \geqq d_n^+(\alpha)\}$, if this limit exists. Now it is well known that as $n \to \infty$ the stochastic process $Z_n(t)$ is asymptotically a certain normal stochastic process $Z(t)$, and that for each $\alpha > 0$ the critical values $d_n(\alpha)$ and $d_n^+(\alpha)$, respectively, approach limits $d(\alpha)$ and $d^+(\alpha)$ determined by

$$\alpha = 2 \textstyle\sum_{k=1}^{\infty} (-1)^{k-1} \exp(-2k^2 d^2) = \exp[-2(d^+)^2]. \tag{1.2}$$

Suppose that the sequence of functions $\{C_n\}$ also tends to some limit, say C. It was Doob's conjecture, later justified by Donsker [6] under fairly general conditions, that in calculating asymptotic $Z_n(t)$ process distributions when $n \to \infty$ we may simply replace the $Z_n(t)$ processes by the $Z(t)$ process. Extending this "heuristic procedure" to the case at hand, we would conclude that

$$P(\alpha, \{C_n\}) = P\{\sup_{0<t<1} |Z(t) - C(t)| \geqq d(\alpha)\}$$

and

$$P^+(\alpha, \{C_n\}) = P\{\sup_{0<t<1} [Z(t) - C(t)] \geqq d^+(\alpha)\}.$$

In the remainder of this paper we investigate the validity of this extended heuristic procedure, and we use the results to establish various bounds on the asymptotic power functions of the two tests.

2. The function $Q(y; A, B)$. We shall say that a function C, defined on the open interval (0, 1), is *piecewise-continuous* if it has at most countably many points of discontinuity. We permit C to take on infinite values, where, if $C(t_0) = \pm\infty$, then we say that t_0 is a point of continuity of C if and only if $C(t) = C(t_0)$ for all t in some neighborhood of t_0. Then if the two functions A and B are both piecewise-continuous, we may define

$$Q(y; A, B) = P\{A(t) - y \leqq Z(t) \leqq B(t) + y, 0 < t < 1\}. \tag{2.1}$$

This probability is legitimate, since if $T = \{t_1, t_2, \cdots\}$ is any countable dense subset of (0, 1) which includes all the points of discontinuity of A and B, and if $T_k = \{t_1, t_2, \cdots, t_k\}$, then it is a simple consequence of separability (which we shall assume) that $Q(y; A, B) = \lim_{k\to\infty} P\{A(t) - y \leqq Z(t) \leqq B(t) + y, t \in T_k\}$. It will be convenient to define also

$$y^*[A, B] = \max\{A(0), A(1), -B(0), -B(1)\} \tag{2.2}$$

where $A(0) = \limsup_{t\to 0^+} A(t)$, $A(1) = \limsup_{t\to 1^-} A(t)$, $B(0) = \liminf_{t\to 0^+} B(t)$, and $B(1) = \liminf_{t\to 1^-} B(t)$. If A and B are sufficiently clear from the context we may drop them from the notation and write only $Q(y)$ and y^*.

The following properties of the $Z(t)$ process may be found in Doob [7] or derived immediately from results therein. First, $Z(t)$ has mean value function $E[Z(t)] = 0$ and covariance kernel cov $[Z(s), Z(t)] = s(1 - t)$ for $0 \leqq s \leqq t \leqq 1$, and $Z(0) = Z(1) = 0$ with probability 1. Hence if $Z_i = Z(t_i)$, $1 \leqq i \leqq k$, where $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$, then the joint probability density function of the random variables $Z_1, Z_2, \cdots, Z_k$ is

$$(2.3) \qquad \psi(z_1, z_2, \cdots, z_k) = \frac{\exp\{-\frac{1}{2}\sum_{i=1}^{k+1}(z_i - z_{i-1})^2/(t_i - t_{i-1})\}}{(2\pi)^{k/2}[t_1(t_2-t_1)\cdots(1-t_k)]^{\frac{1}{2}}}$$

where $z_0 \equiv z_{k+1} \equiv 0$. The $Z(t)$ process can be converted into the classical Wiener process, or Brownian motion, by the transformation

$$(2.4) \qquad W(\tau) = (\tau + 1)Z(\tau + 1)^{-1}, \qquad 0 \leqq \tau < \infty;$$

then we have $E[W(\tau)] = 0$ and cov $[W(\sigma), W(\tau)] = \sigma$ for $0 \leqq \sigma \leqq \tau < \infty$. Since, as is well known, the sample functions of the Wiener process are continuous with probability 1, this property must hold also for $Z(t)$. For future reference it will be convenient to convert two of Doob's results about $W(\tau)$ into results about $Z(t)$. First, since

$$P\{-\alpha\tau - \beta \leqq W(\tau) \leqq a\tau + b, 0 < \tau < \infty\}$$
$$= P\{-\beta t - \alpha(1 - t) \leqq Z(t) \leqq bt + a(1 - t), 0 < t < 1\}$$

we have by his formula (4.3) that if $a, \alpha, b, \beta \geqq 0$ then

$$(2.5) \quad P\{-\beta t - \alpha(1 - t) \leqq Z(t) \leqq bt + a(1 - t), 0 < t < 1\}$$
$$= 1 - \sum_{k=1}^{\infty} e^{-K}\{e^{2(2k-1)ab} + e^{2(2k-1)\alpha\beta} - e^{-4ka\beta} - e^{-4k\alpha b}\}$$

where

$$K = 2k^2(a + \alpha)(b + \beta) - 2k(a\beta + \alpha b);$$

and similarly, using his formula (4.2), we have that for $a, b \geqq 0$

$$(2.6) \qquad P\{Z(t) \leqq bt + a(1 - t), 0 < t < 1\} = 1 - e^{-2ab}.$$

We note that (1.2) may be obtained as a special case of (2.5) and (2.6).

It is convenient at this point to define the function

$$R(t) = [t(1 - t)]^{-\frac{1}{2}}.$$

LEMMA 2.1. *If* $0 < \delta < 1$ *and* $\lambda > 0$ *then*

$$P\{\sup_{0<t<\delta} Z(t) > \lambda\} = \Phi(-\lambda R(\delta)) + \exp(-2\lambda^2)\Phi(-\lambda(1 - 2\delta)R(\delta)) \leqq \delta\lambda^{-2},$$

where Φ *is the standard normal distribution function.*

PROOF. Let P be the desired probability; then

$$P = 1 - P\{Z(t) \leqq \lambda, 0 < t < \delta\}$$
$$= 1 - \int_{-\infty}^{\lambda} P\{Z(t) \leqq \lambda,\ 0 < t < \delta \mid Z(\delta) = z\}\psi(z)\, dz,$$

where, by (2.3), $\psi(z) = (2\pi)^{-\frac{1}{2}}R(\delta) \exp[-z^2R^2(\delta)/2]$. Now if $0 \leqq s < t \leqq 1$, and

$$U(\tau) = (t - s)^{-\frac{1}{2}}\{Z(\tau s + [1 - \tau]t) - \tau Z(s) - (1 - \tau)Z(t)\}, \quad 0 \leqq \tau \leqq 1, \tag{2.7}$$

it may easily be verified that $E[U(\tau)] = 0$ and $\operatorname{cov}[U(\sigma), U(\tau)] = \sigma(1 - \tau)$ for $0 \leqq \sigma \leqq \tau \leqq 1$; thus (2.7) converts the $Z(t)$ process into itself. Using this transformation, with $s = 0$ and $t = \delta$, we find that

$$P\{Z(t) \leqq \lambda, 0 < t < \delta \mid Z(\delta) = z\}$$
$$= P\{\delta^{\frac{1}{2}}Z(t) \leqq \lambda t + (\lambda - z)(1 - t), 0 < t < 1\}$$
$$= 1 - e^{-2\lambda(\lambda - z)/\delta}$$

by (2.6). The stated formula for P then follows by straightforward integration. Now the following inequalities are well known: (a) $2\Phi(-x) \leqq \exp(-x^2/2)$ for $x > 0$; (b) $e^{-x} \leqq 1/xe$ for $x > 0$; (c) $\Phi(x) \leqq 1$ for all x. That the first term of the formula is not greater than $\delta/2\lambda^2$ may be shown using (a) and then (b); that the second term is also not greater than $\delta/2\lambda^2$ may be shown using (a) and then (b) if $\delta < \frac{1}{2}$, or (c) and then (b) if $\delta \geqq \frac{1}{2}$.

LEMMA 2.2. *Let B be any function of t, and let S be any countable subset of $(\delta, 1 - \delta)$, where $0 < \delta < \frac{1}{2}$. Then $P\{Z(t) \leqq B(t) + y, t \varepsilon S\}$ is a monotonically increasing and differentiable function of y, its derivative being nowhere greater than $2R(\delta)$.*

PROOF. Let $T = \{t_i, 1 \leqq i \leqq k\}$ be any finite subset of $(\delta, 1 - \delta)$, where $0 = t_0 < \delta < t_1 < t_2 < \cdots < t_k < 1 - \delta < t_{k+1} = 1$. Write $B(t_i) = B_i$, $1 \leqq i \leqq k$.
Then we have

$$P\{Z(t) \leqq B(t) + y, t \varepsilon T\} = \int_{-\infty}^{B_1+y} \cdots \int_{-\infty}^{B_k+y} \psi(z_1, z_2, \cdots, z_k)\, dz_k \cdots dz_1,$$

where $\psi(z_1, z_2, \cdots, z_k)$ is defined by (2.3). Differentiating with respect to y, we obtain

$$(d/dy)P\{Z(t) \leqq B(t) + y, t \varepsilon T\}$$
$$= -\int_{-\infty}^{B_1+y} \cdots \int_{-\infty}^{B_k+y} [z_1/t_1 + z_k/(1 - t_k)]\psi(z_1, z_2, \cdots, z_k)\, dz_k \cdots dz_1$$

which is less than the expected value of $|z_1/t_1 + z_k/(1 - t_k)|$, and it is easily verified that $(z_1/t_1 + z_k/(1 - t_k))$ is normally distributed with mean 0 and variance $(1/t_1 + 1/(1 - t_k))$. Hence the derivative is less than $[(2/\pi)(1/t_1 + 1/(1 - t_k))]^{\frac{1}{2}}$, which is less than $2R(\delta)$ since $\delta < t_1 < t_k < 1 - \delta$. Now let S_k be the finite set consisting of the first k members of S; then for every k we have $P\{Z(t) \leqq B(t) + y + \epsilon, t \varepsilon S_k\} - P\{Z(t) \leqq B(t) + y - \epsilon, t \varepsilon S_k\} \leqq 4\epsilon R(\delta)$.

We then obtain the lemma if we first let $k \to \infty$ and then divide both sides by 2ϵ and let $\epsilon \to 0$.

THEOREM 2.3. *Let A and B be piecewise-continuous: then*

(a) *$Q(y; A, B)$ is a continuous function of y except possibly at $y = y^*[A, B]$ as defined by* (2.2);

(b) *$Q(y; A, B) = 0$ if $y < y^*[A, B]$;*

(c) *if* $\limsup_{t \to 0^+} \dfrac{y^*[A, B] - \max\{A(t), A(1-t), -B(t), -B(1-t)\}}{[2t(1-t)\log\log((1-t)/t)]^{\frac{1}{2}}} < 1,$

then $Q(y^*; A, B) = 0$ and hence $Q(y; A, B)$ is continuous for all y.

PROOF. Consider first the case where $y < y^*$: say $y = y^* - \epsilon$ for some $\epsilon > 0$. Without loss of generality we may suppose that $y^* = \limsup_{t \to 0^+} A(t)$, so that $\limsup_{t \to 0^+} A(t) - y = \epsilon$. Then

$$\begin{aligned} Q(y) &= P\{A(t) - y \leqq Z(t) \leqq B(t) + y, 0 < t < 1\} \\ &\leqq P\{\limsup_{t \to 0^+} A(t) - y \leqq \limsup_{t \to 0^+} Z(t)\} \\ &= P\{\limsup_{t \to 0^+} Z(t) \geqq \epsilon\} \\ &= 0 \end{aligned}$$

since, with probability 1, $Z(t)$ is continuous in t and $Z(0) = 0$. Thus we have (b).

Now consider the case where $y > y^*$: say $y = y^* + 2\epsilon$ for $\epsilon > 0$. Then there is a $\delta_0(\epsilon)$ sufficiently small that for $0 < \delta < \delta_0(\epsilon)$ we have

$$\inf_{0<t<\delta} B(t) \geqq \liminf_{t \to 0^+} B(t) - \epsilon \geqq -y^*[A, B] - \epsilon = -y + \epsilon.$$

and

$$\inf_{1-\delta<t<1} B(t) \geqq \liminf_{t \to 0^+} B(1-t) - \epsilon \geqq -y^*[A, B] - \epsilon = -y + \epsilon.$$

Let T be a countable dense subset of $(0, 1)$ which includes all the points of discontinuity of B, and let $T_\delta = T \cap (\delta, 1 - \delta)$. Then

$$\begin{aligned} P\{Z(t) &\leqq B(t) + y - \epsilon\delta, T \varepsilon T_\delta\} - P\{Z(t) \leqq B(t) + y - \epsilon\delta, t \varepsilon T\} \\ &= P\{Z(t) \leqq B(t) + y - \epsilon\delta, T \varepsilon T_\delta, \text{ but } Z(t) > B(t) + y - \epsilon\delta, \\ &\qquad \text{for some } t \varepsilon T\} \\ &\leqq P\{Z(t) > B(t) + y - \epsilon\delta \text{ for some } t, 0 < t < \delta\} \\ &\qquad + P\{Z(t) > B(t) + y - \epsilon\delta \text{ for some } t, 1 - \delta < t < 1\} \\ &\leqq P\{\sup_{0<t<\delta} Z(t) > \inf_{0<t<\delta} B(t) + y - \epsilon\delta\} \\ &\qquad + P\{\sup_{1-\delta<t<1} Z(t) > \inf_{1-\delta<t<1} B(t) + y - \epsilon\delta\} \\ &\leqq P\{\sup_{0<t<\delta} Z(t) > \epsilon(1-\delta)\} + P\{\sup_{1-\delta<t<1} Z(t) > \epsilon(1-\delta)\}. \end{aligned}$$

Hence, noting that the processes $Z(t)$ and $Z(1 - t)$ are identical, by Lemma 2.1 we have $P\{Z(t) \leqq B(t) + y - \epsilon\delta, T \varepsilon T_\delta\} - P\{Z(t) \leqq B(t) + y - \epsilon\delta,$

$t \varepsilon T\} \leqq 2\delta(1 - \delta)^{-2}\epsilon^{-2}$. By Lemma 2.2 we have $P\{Z(t) \leqq B(t) + y + \epsilon\delta, t \varepsilon T_\delta\} - P\{Z(t) \leqq B(t) + y - \epsilon\delta, t \varepsilon T_\delta\} \leqq 4\epsilon\delta R(\delta)$ and certainly $P\{Z(t) \leqq B(t) + y + \epsilon\delta, t \varepsilon T\} - P\{Z(t) \leqq B(t) + y + \epsilon\delta, t \varepsilon T_\delta\} \leqq 0$. Adding these last three inequalities, we have

$$P\{Z(t) \leqq B(t) + y + \epsilon\delta, t \varepsilon T\} - P\{Z(t) \leqq B(t) + y - \epsilon\delta, t \varepsilon T\} \leqq 2\delta(1 - \delta)^{-2}\epsilon^{-2} + 4\epsilon\delta R(\delta).$$

On letting $\delta \to 0$, we see that $P\{Z(t) \leqq B(t) + y, t \varepsilon T\}$ is continuous for $y > y^*$, and by separability this is the same as $P\{Z(t) \leqq B(t) + y, 0 < t < 1\}$, or $Q(y; -\infty, B)$. A similar argument will show that $Q(y; A, +\infty)$ is continuous for $y > y^*$. Now $Q(y; A, +\infty)$ and $Q(y; -\infty, B)$ may be regarded as the distribution functions of $Y_1 = \sup_{0<t<1} [A(t) - Z(t)]$ and $Y_2 = \sup_{0<t<1} [Z(t) - B(t)]$ respectively. Since both these distribution functions are continuous, the distribution function $Q(y; A, B)$ of $Y = \max(Y_1, Y_2)$ must also be continuous for $y > y^*[A, B]$, and thus (a) is proven.

In proving (c) we write $f(t) = [2t(1 - t) \log\log((1 - t)/t)]^{\frac{1}{2}}$. By hypothesis $\limsup_{t\to 0^+} [(y^*[A, B] - \max\{A(t), A(1 - t), -B(t), -B(1 - t)\})/f(t)] < 1$. Assuming, without loss of generality, that $y^*[A, B] = -\liminf_{t\to 0^+} B(t)$, we have that $\limsup_{t\to 0^+} \{[y^* + B(t)]/f(t)\} = 1 - 2\epsilon$ for some $\epsilon > 0$. Then there is a $\delta(\epsilon)$ sufficiently small that for $0 < t < \delta(\epsilon)$ we have $y^* + B(t) < (1 - \epsilon)f(t)$, and hence

$$\begin{aligned} Q(y) &= P\{A(t) - y^* \leqq Z(t) \leqq B(t) + y^*, 0 < t < 1\} \\ &\leqq P\{Z(t) \leqq B(t) + y^*, 0 < t < \delta\} \\ &\leqq P\{Z(t) \leqq (1 - \epsilon)f(t), 0 < t < \delta\} \\ &\leqq P\{\limsup_{t\to 0^+} [Z(t)/f(t)] \leqq 1 - \epsilon\} \\ &= 0 \end{aligned}$$

since Khinchin's law of the iterated logarithm, as given (for example) in formula (73), page 242, of Lévy [10], states that $P\{\limsup_{\tau\to\infty} [\xi(\tau)/[2 \log\log \tau]^{\frac{1}{2}}] = 1\} = 1$, where $\tau^{\frac{1}{2}}\xi(\tau)$ is the Wiener process, and by (2.4) this is equivalent to $P\{\limsup_{t\to 0^+} [Z(t)/f(t)] = 1\} = 1$.

3. Justification of the extended heuristic procedure. If A and B are any two functions defined on (0, 1) we may define, in analogy with (2.1),

$$Q_n(y; A, B) = P\{A(t) - y \leqq Z_n(t) \leqq B(t) + y, 0 < t < 1\}.$$

This probability is legitimate, since it can be shown without difficulty that the event concerned is equivalent to the event that each observation of the ordered random sample lies in a certain well-defined interval.

The starting point for our investigation into the validity of the extended heuristic procedure is the work of Donsker in [6]. Specializing the general functional F of his theorem to $F[g] = \sup_{0<t<1} \max\{A(t) - g(t), g(t) - B(t)\}$, we have immediately

THEOREM 3.1. *Let A and B be piecewise-continuous; then*

$$\lim_{n\to\infty} Q_n(y; A, B) = Q(y; A, B)$$

at all points of continuity of $Q(y)$.

Even without using this theorem, it is possible to prove

THEOREM 3.2. *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of functions such that* lim inf A_n *and* lim sup B_n *are piecewise-continuous; then*

$$\limsup_{n\to\infty} Q_n(y; A_n, B_n) \leqq Q(y; \liminf A_n, \limsup B_n).$$

PROOF. Let T be a countable dense subset of $(0, 1)$ which includes all the points of discontinuity of $\liminf A_n$ and $\limsup B_n$; and, for any positive integer k, let T_k be the finite set consisting of the first k members of T. Choosing also any $\epsilon > 0$, there must be an $n_0(\epsilon, k)$ sufficiently large that for $n > n_0(\epsilon, k)$ we have $A_n(t) \geqq \liminf A_n(t) - \epsilon$ and $B_n(t) \leqq \limsup B_n(t) + \epsilon$ for $t \varepsilon T_k$. Then

$$\begin{aligned} Q_n(y; A_n, B_n) &\leqq P\{A_n(t) - y \leqq Z_n(t) \leqq B_n(t) + y, t \varepsilon T_k\} \\ &\leqq P\{\liminf A_n(t) - \epsilon - y \leqq Z_n(t) \leqq \limsup B_n(t) \\ &\qquad + \epsilon + y, t \varepsilon T_k\}. \end{aligned}$$

Letting n tend to infinity, we have $\limsup_{n\to\infty} Q_n(y; A_n, B_n) \leqq P\{\liminf A_n(t) - \epsilon - y \leqq Z(t) \leqq \limsup B_n(t) + \epsilon + y, t \varepsilon T_k\}$. Since ϵ and k were arbitrary, we may now let $\epsilon \to 0$ and then let $k \to \infty$, which yields the theorem.

Combining the preceding two theorems, we can obtain

THEOREM 3.3. *Let $\{A_n\}$ and $\{B_n\}$ be two sequences of functions which converge to piecewise-continuous functions A and B respectively; and suppose also that*

$$\lim_{n\to\infty} \sup_{0<t<1} \max \{A_n(t) - A(t), B(t) - B_n(t)\} = 0.$$

Then $\lim_{n\to\infty} Q_n(y; A_n, B_n) = Q(y; A, B)$ at all points of continuity of $Q(y)$.

PROOF. Since $\limsup_{n\to\infty} Q_n(y; A_n, B_n) \leqq Q(y; A, B)$ by Theorem 3.2, we need show only that $\liminf_{n\to\infty} Q_n(y; A_n, B_n) \geqq Q(y; A, B)$. Choose any $\epsilon > 0$; then by hypothesis there is an $n_0(\epsilon)$ sufficiently large that for $n > n_0(\epsilon)$ we have $\sup_{0<t<1} \max \{A_n(t) - A(t), B(t) - B_n(t)\} < \epsilon$. Thus $A_n(t) < A(t) + \epsilon$ and $B_n(t) > B(t) - \epsilon$ for $0 < t < 1$, so that $Q_n(y; A_n, B_n) \geqq Q_n(y - \epsilon; A, B)$, for $n > n_0(\epsilon)$. Letting n tend to infinity, we find that $\liminf_{n\to\infty} Q_n(y; A_n, B_n) \geqq \liminf_{n\to\infty} Q_n(y - \epsilon; A, B) = Q(y - \epsilon; A, B)$ by Theorem 3.1. Then we can obtain the present theorem on letting $\epsilon \to 0$.

In order to extend these results further we require the following

LEMMA 3.4. *If $0 < \tau - \delta < \tau + \delta < 1$, $\epsilon \geqq 0$, and X is arbitrary, then*

$$P\{\inf_{t\varepsilon I} Z_n(t) \leqq X \leqq \sup_{t\varepsilon I} Z_n(t) + \epsilon\} \leqq \xi[\delta^{\frac{1}{3}} + \epsilon + n^{-\frac{1}{2}}]R(\tau)$$

and

$$P\{\inf_{t\varepsilon I} Z(t) \leqq X \leqq \sup_{t\varepsilon I} Z(t) + \epsilon\} \leqq \xi[\delta^{\frac{1}{3}} + \epsilon]R(\tau),$$

where $I = [\tau - \delta, \tau + \delta]$ and ξ is a finite positive constant.

PROOF. We need prove only the first part of the lemma, since the second will follow immediately by Donsker's theorem. So let E be the event whose probability is to be bounded, and let A be the event that $\sup_{t\varepsilon I} Z_n(t) - \inf_{t\varepsilon I} Z_n(t) > \delta^{\frac{1}{3}}$. Then $P\{E\} = P\{E \cap A\} + P\{E \cap \bar{A}\} \leqq P\{A\} + P\{X - \epsilon - \delta^{\frac{1}{3}} \leqq Z_n(\tau) \leqq X + \delta^{\frac{1}{3}}\}$. Now this latter probability is, using the definition of the $Z_n(t)$ process,

$$P\{(X - \epsilon - \delta^{\frac{1}{3}})/[\tau(1 - \tau)]^{\frac{1}{2}} \leqq (nF_n(\tau) - n\tau)/[n\tau(1 - \tau)]^{\frac{1}{2}} \leqq (X + \delta^{\frac{1}{3}})/[\tau(1-\tau)]^{\frac{1}{2}}\},$$

where $nF_n(\tau)$ is a binomial random variable with parameters (n, τ). Hence by the Berry-Esseen uniform central limit theorem [1], this probability is not greater than $\{\Phi([X + \delta^{\frac{1}{3}}]R(\tau)) - \Phi([X - \epsilon - \delta^{\frac{1}{3}}]R(\tau))\} + \theta n^{-\frac{1}{2}}[\tau^2 + (1 - \tau)^2]R(\tau)$, where θ is an absolute constant and Φ is the standard normal distribution function. The derivative of Φ is everywhere less than $(2\pi)^{-\frac{1}{2}} < 1$, so the first term above is less than $(2\delta^{\frac{1}{3}} + \epsilon)R(\tau)$; and $[\tau^2 + (1 - \tau)^2] < 1$, so the second term is less than $\theta n^{-\frac{1}{2}}R(\tau)$; thus we have $P\{E\} \leqq P\{A\} + (2\delta^{\frac{1}{3}} + \epsilon + \theta n^{-\frac{1}{2}})R(\tau)$. Now let B be the event that $|Z_n(\tau - \delta) - Z_n(\tau + \delta)| \geqq \delta^{\frac{1}{3}}/3$, and let C be the event that $\sup_{t\varepsilon I} |Z_n(t) - Z_n(\tau - \delta) - [(t - \tau + \delta)/2\delta][Z_n(\tau + \delta) - Z_n(\tau - \delta)]| > \delta^{\frac{1}{3}}/3$. Then A cannot occur unless either B or C occurs, so that $P\{A\} \leqq P\{B\} + P\{C\}$. Since the variance of $[Z_n(\tau - \delta) - Z_n(\tau + \delta)]$ is $2\delta(1 - 2\delta) < 2\delta$ we have by the Chebyshev inequality that $P\{B\} \leqq 18\delta^{\frac{1}{3}}$. Let α_{ij} be the conditional probability that C occurs, given that $F_n(\tau - \delta) = i/n$ and $F_n(\tau + \delta) = j/n$; and let β_{ij} be the unconditional probability of this latter event, so that

$$\beta_{ij} = [n!/i!(j - i)!(n - j)!](\tau - \delta)^i(2\delta)^{j-i}(1 - \tau - \delta)^{n-j}, \quad 0 \leqq i \leqq j \leqq n.$$

Now, given that $F_n(\tau - \delta) = i/n$ and $F_n(\tau + \delta) = j/n$, that portion of the empirical distribution function $F_n(t)$ where $\tau - \delta \leqq t \leqq \tau + \delta$ is itself, except for a linear transformation, an empirical distribution function: namely, if we let $F^*_{j-i}(u) = (nF_n(t) - i)/(j - i)$, where $u = (t - \tau + \delta)/2\delta$, then F^*_{j-i} is the empirical distribution function of a random sample of size $(j - i)$. Using this transformation, we find after some simplification that

$$\alpha_{ij} = P\{\sup_{0<u<1} |F^*_{j-i}(u) - u| > \delta^{\frac{1}{3}}n^{\frac{1}{2}}/3(j - i)\},$$
$$\leqq c \exp[-2\delta^{2/3}n^2/9(j - i)^2],$$

where c is the finite positive constant of Lemma 2 of Dvoretzky, Kiefer, and Wolfowitz [8]. Hence $\alpha_{ij} < 9c(j - i)^2/2e\delta^{2/3}n^2 < 9c(j - i)/\delta^{2/3}n$ and $P\{C\} = \sum\sum \alpha_{ij}\beta_{ij} < 9c\delta^{-2/3}n^{-1} \sum\sum (j - i)\beta_{ij} = 18c\delta^{\frac{1}{3}}$. Then $P\{A\} \leqq 18(c + 1)\delta^{\frac{1}{3}} \leqq 9(c + 1)\delta^{\frac{1}{3}}R(\tau)$ since $R(\tau) \geqq 2$ for $0 < \tau < 1$; and hence $P\{E\} \leqq [(9c + 11)\delta^{\frac{1}{3}} + \epsilon + \theta n^{-\frac{1}{2}}]R(\tau)$. The lemma then follows if we take $\xi = \max(9c + 11, \theta)$.

Given any $t, \delta \geqq 0$, define $I(t, \delta) = (t - \delta, t + \delta) \cap (0, 1)$; then we have

THEOREM 3.5. *Let the two sequences of functions $\{A_n\}$ and $\{B_n\}$ be such that the two functions*

$$A(t) = \liminf_{\delta\to 0} \liminf_{n\to\infty} \sup_{\tau\varepsilon I(t,\delta)} A_n(\tau)$$

and

$$B(t) = \limsup_{\delta\to 0} \limsup_{n\to\infty} \inf_{\tau\varepsilon I(t,\delta)} B_n(\tau)$$

are piecewise-continuous; then

$$\limsup_{n\to\infty} Q_n(y; A_n, B_n) \leqq Q(y; A, B).$$

PROOF. Let $T = \{t_1, t_2, \cdots\}$ be a countable dense subset of (0, 1) which includes all the points of discontinuity of A and B, and for $i = 1, 2, \cdots$ and $\delta > 0$ let $I_i = I(t_i, \delta)$. If E_n is the event that $A_n(t) - y \leqq Z_n(t) \leqq B_n(t) + y$ for $0 < t < 1$, then certainly, for any events U_n and V_n,

$$Q_n(y; A_n, B_n) = P\{E_n\} \leqq P\{E_n \cap U_n\} + P\{E_n \cap V_n\} + P\{\bar{U}_n \cap \bar{V}_n\}$$

and hence

$$\limsup_{n\to\infty} Q_n(y; A_n, B_n) \leqq \limsup_{n\to\infty} P\{E_n \cap U_n\} + \limsup_{n\to\infty} P\{E_n \cap V_n\} + \limsup_{n\to\infty} P\{\bar{U}_n \cap \bar{V}_n\}.$$

So, for each positive integer k, choose a $\delta > 0$ sufficiently small that $I_i \subset (0, 1)$ for $1 \leqq i \leqq k$, let $U_n = U_n(k, \delta)$ be the event that $\inf_{t\varepsilon I_i} Z_n(t) < A(t_i) - y$ for some i, $1 \leqq i \leqq k$, and let $V_n = V_n(k, \delta)$ be the event that $\sup_{t\varepsilon I_i} Z_n(t) > B(t_i) + y$ for some i, $1 \leqq i \leqq k$. Then

$$\begin{aligned}
P\{E_n \cap U_n\} &\leqq \textstyle\sum_{i=1}^{k} P\{A_n(t) - y \leqq Z_n(t) \leqq B_n(t) + y,\ 0 < t < 1; \\
&\qquad\qquad \text{but } \inf_{t\varepsilon I_i} Z_n(t) \leqq A(t_i) - y\} \\
&\leqq \textstyle\sum_{i=1}^{k} P\{\sup_{t\varepsilon I_i} A_n(t) - y \leqq \sup_{t\varepsilon I_i} Z_n(t), \\
&\qquad\qquad \inf_{t\varepsilon I_i} Z_n(t) \leqq A(t_i) - y\} \\
&\leqq \textstyle\sum_{i=1}^{k} P\{\inf_{t\varepsilon I_i} Z_n(t) \leqq A(t_i) - y \leqq \sup_{t\varepsilon I_i} Z_n(t) \\
&\qquad\qquad + A(t_i) - \sup_{t\varepsilon I_i} A_n(t)\} \\
&\leqq \xi \textstyle\sum_{i=1}^{k} [\delta^{\frac{1}{4}} + \max\{A(t_i) - \sup_{t\varepsilon I_i} A_n(t),\ 0\} + n^{-\frac{1}{4}}]R(t_i),
\end{aligned}$$

where ξ is the finite positive constant of Lemma 3.4. Hence

$$\limsup_{n\to\infty} P\{E_n \cap U_n\} \leqq \xi \textstyle\sum_{i=1}^{k} [\delta^{\frac{1}{4}} + \max\{A(t_i) - \liminf_{n\to\infty} \sup_{t\varepsilon I_i} A_n(t), 0\}]R(t_i).$$

But it is obvious that $A(t_i) \leqq \liminf_{n\to\infty} \sup_{t\varepsilon I_i} A_n(t)$, and thus

$$\limsup_{n\to\infty} P\{E_n \cap U_n\} \leqq \xi\delta^{\frac{1}{4}} \textstyle\sum_{i=1}^{k} R(t_i).$$

By a similar argument we can obtain the same upper bound for $\limsup_{n\to\infty} P\{E_n \cap V_n\}$. Finally,

$$\begin{aligned}
P\{\bar{U}_n \cap \bar{V}_n\} &= P\{A(t_i) - y \leqq Z_n(t) \leqq B(t_i) + y,\ t \varepsilon I_i,\ 1 < i < k\} \\
&\leqq P\{A(t_i) - y \leqq Z_n(t_i) \leqq B(t_i) + y,\ 1 \leqq i \leqq k\}
\end{aligned}$$

so $\limsup_{n\to\infty} P\{\bar{U}_n \cap \bar{V}_n\} \leqq P\{A(t_i) - y \leqq Z(t_i) \leqq B(t_i) + y, 1 \leqq i \leqq k\}$. Hence $\limsup_{n\to\infty} Q_n(y; A_n, B_n) \leqq P\{A(t_i) - y \leqq Z(t_i) \leqq B(t_i) + y, 1 \leqq i \leqq k\} + 2\xi\delta^{\frac{1}{2}} \sum_{i=1}^k R(t_i)$. Since this is true for every sufficiently small δ, we may let $\delta \to 0$; and then since k was arbitrary we may let $k \to \infty$; which proves the theorem.

THEOREM 3.6. *Given two sequences of functions $\{A_n\}$ and $\{B_n\}$, suppose that* (a) *for $0 < t < 1$ the two limit functions*

$$A(t) = \lim_{\delta\to 0} \lim_{n\to\infty} \sup_{\tau\in I(t,\delta)} A_n(\tau)$$

and

$$B(t) = \lim_{\delta\to 0} \lim_{n\to\infty} \inf_{\tau\in I(t,\delta)} B_n(\tau)$$

exist and are piecewise-continuous; *and* (b) *there is some finite subset $T = \{t_1, t_2, \cdots, t_k\}$ of $(0, 1)$ such that for every $\delta > 0$, $\lim_{n\to\infty} \sup_{t\notin J} \max \{A_n(t) - A(t), B(t) - B_n(t)\} = 0$, where $J = \bigcup_{1\leqq i\leqq k} I(t_i, \delta)$. Then $\lim_{n\to\infty} Q_n(y; A_n, B_n) = Q(y; A, B)$ at all points of continuity of $Q(y)$.*

PROOF: Using (a) we have immediately by Theorem 3.5 that

$$\limsup_{n\to\infty} Q_n(y; A_n, B_n) \leqq Q(y; A, B);$$

thus we need prove only that

$$\liminf_{n\to\infty} Q_n(y; A_n, B_n) \geqq Q(y; A, B). \tag{3.1}$$

If $y < y^*[A, B]$, or if $y = y^*$ and y^* is a point of continuity of $Q(y)$, then $Q(y; A, B) = 0$ by Theorem 2.3 and hence (3.1) is trivial. So assume that $y > y^*$: say $y = y^* + 3\lambda$, for some $\lambda > 0$.

Choose any ϵ, $0 < \epsilon < \lambda$; then there must be a $\delta_0(\epsilon)$ sufficiently small that for $0 < \delta < \delta_0(\epsilon)$ we have that $\lim_{n\to\infty} \sup_{t\in I(t_i,\delta)} A_n(t) < A(t_i) + \epsilon$ and

$$\lim_{n\to\infty} \inf_{t\in I(t_i,\delta)} B_n(t) > B(t_i) - \epsilon$$

and that the intervals $I(t_i, \delta)$ are included in $(0, 1)$ and do not overlap, for $1 \leqq i \leqq k$. And there must also be an $n_0(\delta, \epsilon)$ sufficiently large that for $n > n_0(\delta, \epsilon)$ we have

$$\sup_{t\in I(t_i,\delta)} A_n(t) < \lim_{n\to\infty} \sup_{t\in I(t_i,\delta)} A_n(t) + \epsilon < A(t_i) + 2\epsilon$$

and

$$\inf_{t\in I(t_i,\delta)} B_n(t) > \lim_{n\to\infty} \inf_{t\in I(t_i,\delta)} B_n(t) - \epsilon > B(t_i) - 2\epsilon$$

for $1 \leqq i \leqq k$, and also $\sup_{t\notin J} \max \{A_n(t) - A(t), B(t) - B_n(t)\} < 2\epsilon$, that is, $A_n(t) < A(t) + 2\epsilon$ and $B_n(t) > B(t) - 2\epsilon$ for $t \notin J$. Define

$$\begin{aligned} A^{(\delta)}(t) &= A(t_i), \qquad t \in I(t_i, \delta), \qquad 1 \leqq i \leqq k, \\ &= A(t), \qquad \text{otherwise} \end{aligned}$$

and

$$B^{(\delta)}(t) = B(t_i), \qquad t \varepsilon I(t_i, \delta), \qquad 1 \leqq i \leqq k$$
$$= B(t), \qquad \text{otherwise};$$

note that $A^{(\delta)}$ and $B^{(\delta)}$ are piecewise-continuous and that $y^*[A^{(\delta)}, B^{(\delta)}] = y^*[A, B]$. Thus, for any $\epsilon > 0$, any $\delta < \delta_0(\epsilon)$, and any $n > n_0(\delta, \epsilon)$, we have $A_n(t) < A^{(\delta)}(t) + 2\epsilon$ and $B_n(t) > B^{(\delta)}(t) - 2\epsilon$ for $0 < t < 1$; hence $Q_n(y; A_n, B_n) \geqq Q_n(y - 2\epsilon; A^{(\delta)}, B^{(\delta)})$; and, letting n tend to infinity, we have

$$\liminf_{n\to\infty} Q_n(y; A_n, B_n) \geqq \liminf_{n\to\infty} Q_n(y - 2\epsilon; A^{(\delta)}, B^{(\delta)}) = Q(y - 2\epsilon; A^{(\delta)}, B^{(\delta)})$$

by Theorem 3.1. Now define the events

$$E: A(t) - y + 2\epsilon \leqq Z(t) \leqq B(t) + y - 2\epsilon, \qquad 0 < t < 1,$$
$$E_\delta : A^{(\delta)}(t) - y + 2\epsilon \leqq Z(t) \leqq B^{(\delta)}(t) + y - 2\epsilon, \quad 0 < t < 1,$$
$$\alpha_i : A^{(\delta)}(t) - y + 2\epsilon > Z(t) \qquad \text{for some } t \varepsilon I(t_i, \delta),$$
$$\beta_i : Z(t) > B^{(\delta)}(t) + y - 2\epsilon \qquad \text{for some } t \varepsilon I(t_i, \delta).$$

Then

$$Q(y - 2\epsilon; A, B) = P\{E\} = P\{E \cap E_\delta\} + P\{E \cap \bar{E}_\delta\} \leqq Q(y - 2\epsilon; A^{(\delta)}, B^{(\delta)}) + [\textstyle\sum_{i=1}^k P\{E \cap \alpha_i\} + \sum_{i=1}^k P\{E \cap \beta_i\}].$$

Now

$$\begin{aligned}
\mathrm{P}\{E \cap \beta_i\} &\leqq P\{Z(t) \leqq B(t) + y - 2\epsilon, t \varepsilon I(t_i, \delta); \\
&\qquad \text{but } Z(t) > B^{(\delta)}(t) + y - 2\epsilon \text{ for some } t \varepsilon I(t_i, \delta)\} \\
&\leqq P\{\inf_{t\varepsilon I(t_i,\delta)} Z(t) \leqq \inf_{t\varepsilon I(t_i,\delta)} B(t) + y - 2\epsilon, \\
&\qquad \sup_{t\varepsilon I(t_i,\delta)} Z(t) > \inf_{t\varepsilon I(t_i,\delta)} B^{(\delta)}(t) + y - 2\epsilon\} \\
&\leqq P\{\inf_{t\varepsilon I(t_i,\delta)} Z(t) \leqq \inf_{t\varepsilon I(t_i,\delta)} B(t) + y - 2\epsilon \leqq \sup_{t\varepsilon I(t_i,\delta)} Z(t) \\
&\qquad + \inf_{t\varepsilon I(t_i,\delta)} B(t) - \inf_{t\varepsilon I(t_i,\delta)} B^{(\delta)}(t)\}.
\end{aligned}$$

But $\inf_{t\varepsilon I(t_i,\delta)} B^{(\delta)}(t) = B(t_i) \geqq \inf_{t\varepsilon I(t_i,\delta)} B(t)$, so by Lemma 3.4 $P\{E \cap \beta_i\} \leqq \xi\delta^{\frac{1}{2}}R(t_i)$ for $1 \leqq i \leqq k$, and hence $\sum_{i=1}^k P\{E \cap \beta_i\} \leqq \xi\delta^{\frac{1}{2}} \sum_{i=1}^k R(t_i)$. A similar argument produces the same upper bound for $\sum_{i=1}^k P\{E \cap \alpha_i\}$. So $Q(y - 2\epsilon; A, B) \leqq Q(y - 2\epsilon; A^{(\delta)}, B^{(\delta)}) + 2\xi\delta^{\frac{1}{2}} \sum_{i=1}^k R(t_i)$ and hence $\liminf_{n\to\infty} Q_n(y; A_n, B_n) \geqq Q(y - 2\epsilon; A, B) - 2\xi\delta^{\frac{1}{2}} \sum_{i=1}^k R(t_i)$. The theorem then follows if we first let $\delta \to 0$ and then let $\epsilon \to 0$.

REMARK. It is trivial to show that the expression $Q_n(y; A_n, B_n)$ may be replaced by $Q_n(y_n; A_n, B_n)$ where $\lim_{n\to\infty} y_n = y$, in each of the theorems of this section.

4. Bounds on the asymptotic power. Let $\mathcal{C}$ be any class of sequences $\{C_n\}$ of functions C_n as defined by (1.1). Then we define the asymptotic greatest lower bound on the power of the K_n test in $\mathcal{C}$ to be

$$L[\alpha, \mathcal{C}] = \inf_{\{C_n\}\varepsilon\mathcal{C}} \liminf_{n\to\infty} P\{\sup_{0<t<1} |Z_n(t) - C_n(t)| \geqq d_n(\alpha)\},$$

or, equivalently, $L[\alpha, \mathcal{C}] = 1 - \sup_{\{C_n\}\varepsilon\mathcal{C}} \limsup_{n\to\infty} Q_n(d_n ; C_n , C_n)$, where $d_n = d_n(\alpha)$. We also define the asymptotic greatest lower bound on the power of the K_n^+ test in $\mathcal{C}$ to be $L^+[\alpha, \mathcal{C}] = 1 - \sup_{\{C_n\}\varepsilon\mathcal{C}} \limsup_{n\to\infty} Q_n(d_n^+; -\infty, C_n)$, where $d_n^+ = d_n^+(\alpha)$. Asymptotic least upper bounds $U[\alpha, \mathcal{C}]$ and $U^+[\alpha, \mathcal{C}]$ may be defined analogously.

We shall now use the extended heuristic procedure to obtain asymptotic bounds on the power of the two Kolmogorov-Smirnov tests in the following classes of sequences $\{C_n\}$:

$\mathcal{C}_\Delta$, for $\Delta > 0$: All $\{C_n\}$ such that $\lim_{n\to\infty} \sup_{0<t<1} |C_n(t)| = \Delta$;

$\mathcal{C}_\Delta(\tau)$, for $\Delta > 0$ and $0 < \tau < 1$: All $\{C_n\}$ ε $\mathcal{C}_\Delta$ such that

$$\limsup_{\delta\to 0} \limsup_{n\to\infty} \inf_{t\varepsilon I(\tau,\delta)} C_n(t) = -\Delta;$$

(4.1) $\mathcal{C}_\Delta^*$, for $\Delta > 0$: $\bigcup_{0<\tau<1} \mathcal{C}_\Delta(\tau)$;

$\mathcal{C}_\Delta^+(\tau)$, for $\Delta > 0$ and $0 < \tau < 1$: All $\{C_n\}$ ε $\mathcal{C}_\Delta(\tau)$ such that

$$C_n(t) \leqq 0 \text{ for } 0 < t < 1, n = 1, 2, \cdots ;$$

$\mathcal{C}_\Delta^+$, for $\Delta > 0$: $\bigcup_{0<\tau<1} \mathcal{C}_\Delta^+(\tau)$.

The class $\mathcal{C}_\Delta$ corresponds to situations in which the distance between the hypothesis distribution H and the alternative G_n is

$$\Delta_n = \sup_{-\infty<x<\infty} |H(x) - G_n(x)| = \Delta n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}}).$$

In the class $\mathcal{C}_\Delta^+$ we have $G_n(x) \geqq H(x)$ for all x: that is, the hypothesis and alternative are "stochastically comparable" as defined by Birnbaum and Scheuer [3]. It is for such situations that a one-sided test, such as the K_n^+ test, seems most appropriate; however, the K_n^+ test is actually suitable in the larger class $\mathcal{C}_\Delta^*$, where $G_n(x)$ may be less than $H(x)$ for some values of x. The classes $\mathcal{C}_\Delta^+(\tau)$ and $\mathcal{C}_\Delta(\tau)$, which may perhaps be of little interest per se, correspond very roughly to situations in which $G_n(x) = H(x) + \Delta n^{-\frac{1}{2}} + o(n^{-\frac{1}{2}})$, where $x = H^{-1}(\tau)$ for some specified τ.

It should be noted that not all the bounds of this section are entirely new. Various authors have considered the power of the Kolmogorov-Smirnov tests in the case where the sample size increases while the alternative distribution remains fixed. In particular, Birnbaum [2] and Chapman [4] have derived asymptotic expressions equivalent to our results for the K_n^+ test in $\mathcal{C}_\Delta$, $\mathcal{C}_\Delta^+(\tau)$, and $\mathcal{C}_\Delta^+$; and the lowest bounds of Theorem 4.4(b) and (c) are essentially the well-known lower bounds due to Massey [11].

THEOREM 4.1. *Let $\mathcal{C}$ be any of the classes defined by* (4.1); *then*

$$U^+[\alpha, \mathcal{C}] = \exp[-2(\Delta - d^+)^2] \quad \text{for } \Delta \leqq d^+(\alpha)$$
$$= 1 \quad \text{for } \Delta \geqq d^+(\alpha).$$

PROOF. If $\{C_n\}$ is any member of $\mathfrak{C}$, define $B_n(t) = -\sup_{0<t<1} |C_n(t)|$, $0 < t < 1$, so that $\{B_n(t)\}$ converges to $B(t) = -\Delta$, $0 < t < 1$; then $C_n(t) \geqq B_n(t)$, so that $Q_n(d_n^+; -\infty, C_n) \geqq Q_n(d_n^+; -\infty, B_n)$, and thus

$$\inf_{\{C_n\}\varepsilon\mathfrak{C}} \liminf_{n\to\infty} Q_n(d_n^+; -\infty, C_n) \geqq \liminf_{n\to\infty} Q_n(d_n^+; -\infty, B_n) = Q(d^+; -\infty, B)$$

by Theorem 3.3. On the other hand, if

$$\begin{aligned} B_n^*(t) &= -tn^{\frac{1}{2}} & 0 < t \leqq \Delta/n^{\frac{1}{2}} \\ &= -\Delta & \Delta/n^{\frac{1}{2}} \leqq t < 1, \end{aligned}$$

then $\{B_n^*\}$ also converges to B, and $\{B_n^*\}$ is a member of $\mathfrak{C}$, so

$$\inf_{\{C_n\}\varepsilon\mathfrak{C}} \liminf_{n\to\infty} Q_n(d_n^+; -\infty, C_n) \leqq \liminf_{n\to\infty} Q_n(d_n^+; -\infty, B_n^*) = Q(d^+; -\infty, B)$$

by Theorem 3.3 again. Hence

$$U^+[\alpha, \mathfrak{C}] = 1 - \inf_{\{C_n\}\varepsilon\mathfrak{C}} \liminf_{n\to\infty} Q_n(d_n^+; -\infty, C_n) = 1 - Q(d^+; -\infty, B);$$

and the explicit evaluation follows from (2.6).

THEOREM 4.2. *Writing d^+ for $d^+(\alpha)$, we have*

(a) $L^+[\alpha, \mathfrak{C}_\Delta] = \exp[-2(\Delta + d^+)^2]$;

(b) $L^+[\alpha, \mathfrak{C}_\Delta(\tau)]$

$$\begin{aligned} &= \Phi((\Delta - d^+)/[\tau(1-\tau)]^{\frac{1}{2}}) - \Phi((-3\Delta - d^+)/[\tau(1-\tau)]^{\frac{1}{2}}) \\ &\quad + \exp[-2(\Delta + d^+)^2]\{\Phi([-\Delta(1+2\tau) + d^+(1-2\tau)]/[\tau(1-\tau)]^{\frac{1}{2}}) \\ &\quad + \Phi([-\Delta(3-2\tau) - d^+(1-2\tau)]/[\tau(1-\tau)]^{\frac{1}{2}})\}; \end{aligned}$$

(c) $L^+[\alpha, \mathfrak{C}_\Delta^*] = \inf_{0<\tau<1} L^+[\alpha, \mathfrak{C}_\Delta(\tau)]$

$$\begin{aligned} &= e^{-2(\Delta+d^+)^2} && \textit{for } \Delta < d^+, \\ &= \Phi(2\Delta - 2d^+) + f_1(\Delta, d^+) && \textit{for } \Delta \geqq d^+, \end{aligned}$$

where $0 < f_1 \leqq e^{-2(\Delta+d^+)^2 - 8\Delta^2} \leqq \alpha^8$;

(d) $L^+[\alpha, \mathfrak{C}_\Delta^+(\tau)]$

$$\begin{aligned} &= \Phi((\Delta - d^+)/[\tau(1-\tau)]^{\frac{1}{2}}) - \Phi((-\Delta - d^+)/[\tau(1-\tau)]^{\frac{1}{2}}) \\ &\quad + \alpha\{\Phi([-\Delta + d^+(1-2\tau)]/[\tau(1-\tau)]^{\frac{1}{2}}) \\ &\quad + \Phi([-\Delta - d^+(1-2\tau)]/[\tau(1-\tau)]^{\frac{1}{2}})\}; \end{aligned}$$

(e) $L^+[\alpha, \mathfrak{C}_\Delta^+] = \inf_{0<\tau<1} L^+[\alpha, \mathfrak{C}_\Delta^+(\tau)]$

$$\begin{aligned} &= \alpha && \textit{for } \Delta < d^+, \\ &= \Phi(2\Delta - 2d^+) + f_2(\Delta, d^+) && \textit{for } \Delta \geqq d^+, \end{aligned}$$

where $0 \leqq f_1 \leqq f_2 \leqq \alpha \exp(-2\Delta^2) \leqq \alpha^2$.

PROOF. Given any sequence of functions $\{C_n\}$, define

$$C^*(t) = \limsup_{\delta\to 0} \limsup_{n\to\infty} \inf_{\tau\varepsilon I(t,\delta)} C_n(\tau),$$

and write d_n^+ for $d_n^+(\alpha)$.

In proving (a), let $C^{(a)}(t) = \Delta$, $0 < t < 1$; then for any $\{C_n\} \varepsilon \mathcal{C}_\Delta$ clearly $C^*(t) \leqq C^{(a)}(t)$, $0 < t < 1$, so that by Theorem 3.5,

$$\limsup_{n\to\infty} Q_n(d_n^+; -\infty, C_n) \leqq Q(d^+; -\infty, C^*) \leqq Q(d^+; -\infty, C^{(a)}),$$

and hence $\sup_{\{C_n\}\varepsilon\mathcal{C}_\Delta} \limsup_{n\to\infty} Q_n(d_n^+; -\infty, C_n) \leqq Q(d^+; -\infty, C^{(a)})$. On the other hand, let

$$\begin{aligned} B_n(t) &= \Delta, & 0 < t \leqq 1 - \Delta/n^{\frac{1}{2}}, \\ &= n^{\frac{1}{2}}(1-t), & 1 - \Delta/n^{\frac{1}{2}} \leqq t < 1; \end{aligned}$$

then $Q_n(d_n^+; -\infty, B_n) = Q_n(d_n^+; -\infty, C^{(a)})$ since $Z_n(t) \leqq n^{\frac{1}{2}}(1-t) + d_n^+$ for all t. And $\{B_n\}$ is a member of $\mathcal{C}_\Delta$, so

$$\begin{aligned} \sup_{\{C_n\}\varepsilon\mathcal{C}_\Delta} \limsup_{n\to\infty} Q_n(d_n^+; -\infty, C_n) &\geqq \limsup_{n\to\infty} Q_n(d_n^+; -\infty, B_n) \\ &= Q(d^+; -\infty, C^{(a)}). \end{aligned}$$

Hence $L^+[\alpha, \mathcal{C}_\Delta] = 1 - Q(d^+; -\infty, C^{(a)})$, and the explicit evaluation follows from (2.6).

In proving (b), let $C^{(b)}(t) = \Delta$ for $t \neq \tau$, and $C^{(b)}(\tau) = -\Delta$; then for any $\{C_n\} \varepsilon \mathcal{C}_\Delta(\tau)$ clearly $C^*(t) \leqq C^{(b)}(t)$, $0 < t < 1$, so that by Theorem 3.5, $\limsup_{n\to\infty} Q_n(d_n^+; -\infty, C_n) \leqq Q(d^+; -\infty, C^*) \leqq Q(d^+; -\infty, C^{(b)})$, and hence $\sup_{\{C_n\}\varepsilon\mathcal{C}_\Delta(\tau)} \limsup_{n\to\infty} Q_n(d_n^+; -\infty, C_n) \leqq Q(d^+; -\infty, C^{(b)})$. On the other hand, let

$$\begin{aligned} B_n(t) &= \Delta, & 0 < t \leqq \tau - 2\Delta/n^{\frac{1}{2}}, \\ &= n^{\frac{1}{2}}(\tau - t) - \Delta, & \tau - 2\Delta/n^{\frac{1}{2}} \leqq t < \tau, \\ &= \Delta, & \tau \leqq t \leqq 1 - \Delta/n^{\frac{1}{2}}, \\ &= n^{\frac{1}{2}}(1-t), & 1 - \Delta/n^{\frac{1}{2}} \leqq t < 1; \end{aligned}$$

then $\{B_n\}$ is a member of $\mathcal{C}_\Delta(\tau)$ and hence

$$\sup_{\{C_n\}\varepsilon\mathcal{C}_\Delta(\tau)} \limsup_{n\to\infty} Q_n(d_n^+; -\infty, C_n) \geqq \limsup_{n\to\infty} Q_n(d_n^+; -\infty, B_n).$$

In this last expression it will make no difference if we replace $B_n(t)$ by

$$\begin{aligned} B_n^*(t) &= B_n(t), & 0 < t \leqq 1 - \Delta/n^{\frac{1}{2}} \\ &= \Delta, & 1 - \Delta/n^{\frac{1}{2}} \leqq t < 1, \end{aligned}$$

since from its definition we have that $B_n(t) \leqq n^{\frac{1}{2}}(1-t)$ for $0 \leqq t \leqq 1$; and $(\{-\infty\}, \{B_n^*(t)\})$ converges to $(-\infty, C^{(b)})$ in the sense of Theorem 3.6 with $T = \{\tau\}$, so

$$\begin{aligned} \limsup_{n\to\infty} Q_n(d_n^+; -\infty, B_n) &= \limsup_{n\to\infty} Q_n(d_n^+; -\infty, B_n^*) \\ &= Q(d^+; -\infty, C^{(b)}). \end{aligned}$$

Hence $L^+[\alpha, \mathcal{C}_\Delta(\tau)] = 1 - Q(d^+; -\infty, C^{(b)})$, and the explicit evaluation follows by an argument similar to the one used in proving Lemma 2.1.

Since $\mathcal{C}_\Delta{}^* = \bigcup_{0<\tau<1} \mathcal{C}_\Delta(\tau)$ it is clear that $L^+[\alpha, \mathcal{C}_\Delta{}^*] = \inf_{0<\tau<1} L^+[\alpha, \mathcal{C}_\Delta(\tau)$ as stated in (c) above. If $\Delta < d^+$ then $L^+[\alpha, \mathcal{C}_\Delta{}^*] \leqq \lim_{\tau\to 0} L^+[\alpha, \mathcal{C}_\Delta(\tau)] = \exp[-2(\Delta + d^+)^2]$ by (b); but since $\mathcal{C}_\Delta{}^* \subset \mathcal{C}_\Delta$ we have $L^+[\alpha, \mathcal{C}_\Delta{}^*] \geqq L^+[\alpha, \mathcal{C}_\Delta] = \exp[-2(\Delta + d^+)^2]$ by (a). Now suppose that $\Delta \geqq d^+$; then, recalling the definition of $C^{(b)}$, we have $Q(d^+; -\infty, C^{(b)}) \leqq P\{Z(\tau) \leqq -\Delta + d^+) = \Phi((-\Delta + d^+)R(\tau))$ so $L^+[\alpha, \mathcal{C}_\Delta(\tau)] \geqq \Phi((\Delta - d^+)R(\tau))$, and this latter expression is minimized for $\tau = \frac{1}{2}$, so $L^+[\alpha, \mathcal{C}_\Delta(\tau)] \geqq \Phi(2\Delta - 2d^+)$. On the other hand,

$$\begin{aligned} L^+[\alpha, \mathcal{C}_\Delta{}^*] &\leqq L^+[\alpha, \mathcal{C}_\Delta(\tfrac{1}{2})] \\ &= \Phi(2\Delta - 2\, d^+) - \Phi(-6\Delta - 2\, d^+) + 2 \exp[-2(\Delta + d^+)^2]\Phi(-4\Delta), \end{aligned}$$

where $-\Phi(-6\Delta - 2d^+) \leqq 0$ and $\Phi(-4\Delta) \leqq \frac{1}{2}\exp(-8\Delta^2)$, so $L^+[\alpha, \mathcal{C}_\Delta{}^*] \leqq \Phi(2\Delta - 2d^+) + \exp[-2(\Delta + d^+)^2 - 8\Delta^2]$. Since $\exp[-2(d^+)^2] = \alpha$, we see that the proof of (c) is now complete.

In proving (d), let $C^{(d)}(t) = 0$ for $t \neq \tau$, and $C^{(d)}(\tau) = -\Delta$; and let

$$\begin{aligned} B_n(t) &= n^{\frac{1}{2}}(\tau - t) - \Delta, && \tau - \Delta/n \leqq t < \tau, \\ &= 0, && \text{otherwise} \end{aligned} \tag{4.2}$$

and $T = \{\tau\}$; then the result follows by the same argument as for (b).

Since $\mathcal{C}_\Delta{}^+ = \bigcup_{0<\tau<1} \mathcal{C}_\Delta{}^+(\tau)$, it is clear that $L^+[\alpha, \mathcal{C}_\Delta{}^+] = \inf_{0<\tau<1} L^+[\alpha, \mathcal{C}_\Delta{}^+(\tau)]$ as stated in (e) above. Now if $\Delta < d^+$ then $L^+[\alpha, \mathcal{C}_\Delta{}^+] \leqq \lim_{\tau\to 0} L^+[\alpha, \mathcal{C}_\Delta{}^+(\tau)] = \alpha$ by (d); but for any $\{C_n\} \varepsilon\, \mathcal{C}_\Delta{}^+$ we have $C_n(t) \leqq 0$ for $0 < t < 1$ so $\limsup_{n\to\infty} C_n(t) \leqq 0$ and $\limsup_{n\to\infty} Q_n(d_n{}^+; -\infty, C_n) \leqq Q(d; -\infty, 0)$ by Theorem 3.2, and $Q(d; -\infty, 0) = 1 - \alpha$. Hence

$$L^+[\alpha, \mathcal{C}_\Delta{}^+] = 1 - \sup_{\{C_n\}\varepsilon\mathcal{C}_\Delta{}^+} \limsup_{n\to\infty} Q_n(d_n{}^+; -\infty, C_n) \geqq \alpha.$$

Now since $\mathcal{C}_\Delta{}^+ \subset \mathcal{C}_\Delta{}^*$, clearly $L^+[\alpha, \mathcal{C}_\Delta{}^+] \geqq L^+[\alpha, \mathcal{C}_\Delta{}^*]$, while on the other hand

$$\begin{aligned} L^+[\alpha, \mathcal{C}_\Delta{}^+] &\leqq L^+[\alpha, \mathcal{C}_\Delta{}^+(\tfrac{1}{2})] \\ &= \Phi(2\Delta - 2d^+) - \Phi(-2\Delta - 2d^+) + 2\alpha\Phi(-2\Delta) \\ &\leqq \Phi(2\Delta - 2d^+) + \alpha \exp(-2\Delta^2) \end{aligned}$$

by the same argument as for (c); thus the proof of (e) is complete.

THEOREM 4.3.

$$\begin{aligned} \text{(a)}\quad U[\alpha, \mathcal{C}_\Delta] = U[\alpha, \mathcal{C}_\Delta(\tau)] &= U[\alpha, \mathcal{C}_\Delta{}^*] \\ &= 2\textstyle\sum_{k=1}^{\infty} (-1)^{k-1} e^{-2k^2(\Delta-d)^2} && \textit{for } \Delta < d(\alpha) \\ &= 1 && \textit{for } \Delta \geqq d(\alpha) \end{aligned}$$

$$\begin{aligned} \text{(b)}\quad U[\alpha, \mathcal{C}_\Delta{}^+(\tau)] = U[\alpha, \mathcal{C}_\Delta{}^+] &= \textstyle\sum_{k=1}^{\infty} \{e^{-2[k\Delta-(2k-1)d]^2} + e^{-2[(k-1)\Delta-(2k-1)d]^2} \\ &\qquad - 2e^{-2k^2(\Delta-2d)^2}\} && \textit{for } \Delta < d(\alpha), \\ &= 1 && \textit{for } \Delta \geqq d(\alpha). \end{aligned}$$

PROOF. If $\{C_n\}$ is any member of $\mathcal{C}_\Delta$, define $A_n(t) = \sup_{0<t<1} |C_n(t)|$ and $B_n(t) = -A_n(t)$, so that $\{A_n(t)\}$ converges to $A(t) = \Delta$ and $\{B_n(t)\}$ converges to $B(t) = -\Delta$, for $0 < t < 1$. Then $A_n(t) \geqq C_n(t) \geqq B_n(t)$, so that

$$Q_n(d_n ; C_n , C_n) \geqq Q_n(d_n ; A_n , B_n),$$

and thus

$$\inf_{\{C_n\}\varepsilon\mathcal{C}_\Delta} \liminf_{n\to\infty} Q_n(d_n ; C_n , C_n) \geqq \liminf_{n\to\infty} Q_n(d_n ; A_n , B_n) = Q(d; A, B)$$

by Theorem 3.3. On the other hand, if

$$A_n^*(t) = B_n^*(t) = \Delta \sin (n^{\frac{1}{2}}t(1 - t)/\Delta), \qquad 0 < t < 1,$$

then $\{A_n^*\}$ and $\{B_n^*\}$ converge to A and B respectively, in the sense of Theorem 3.6, with T the null set; and furthermore $\{A_n^*\} = \{B_n^*\}$ is a member of $\mathcal{C}_\Delta(\tau)$ for every τ, $0 < \tau < 1$, and hence *a fortiori* of $\mathcal{C}_\Delta^*$ and of $\mathcal{C}_\Delta$; so

$$\inf_{\{C_n\}\varepsilon\mathcal{C}_\Delta} \liminf_{n\to\infty} Q_n(d_n ; C_n , C_n) \leqq \lim_{n\to\infty} Q_n(d_n ; A_n^*, B_n^*) = Q(d; A, B).$$

Hence $U[\alpha, \mathcal{C}_\Delta] = U[\alpha, \mathcal{C}_\Delta^*] = U[\alpha, \mathcal{C}_\Delta(\tau)] = 1 - Q(d; A, B)$ and the explicit evaluation follows from (2.5), giving part (a).

The same argument will serve for part (b) if we take $A_n(t) = 0$, $B_n(t) = -\sup_{0<t<1} |C_n(t)|$, $A(t) = 0$, $B(t) = -\Delta$, and $A_n^*(t) = B_n^*(t) = -\Delta \sin^2 (n^{\frac{1}{2}}t\cdot(1 - t)/2\Delta)$, $0 < t < 1$.

THEOREM 4.4.

(a) *Let $\mathcal{C}$ be any class of sequences $\{C_n\}$ of functions C_n as defined by* (1.1) *and write d for $d(\alpha)$; then*

$$L[\alpha, \mathcal{C}] \geqq L^+[\exp(-2d^2), \mathcal{C}] \geqq L^+[\tfrac{1}{2}\alpha, \mathcal{C}].$$

(b) *For every τ, $0 < \tau < 1$, let*

$$\begin{aligned} B'(t) &= -\Delta, \qquad t = \tau, \\ &= \Delta, \qquad t \neq \tau, \end{aligned}$$

and

$$\begin{aligned} B''(t) &= -\Delta, \qquad t = \tau, \\ &= 0 \qquad t \neq \tau; \end{aligned}$$

then

$$\begin{aligned} \Phi((\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) + \Phi((-\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) &\leqq 1 - Q(d; -\Delta, B') \\ &\leqq L[\alpha, \mathcal{C}_\Delta(\tau)] \\ &\leqq L[\alpha, \mathcal{C}_\Delta^+(\tau)] \\ &\leqq 1 - Q(d; 0, B'') \\ &\leqq \Phi((\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) + \alpha. \end{aligned}$$

(c) *Let*

$$f(\Delta, d) = 0, \qquad \Delta < d(\alpha),$$
$$= \Phi(2\Delta - 2d), \qquad \Delta \geqq d(\alpha);$$

then

$$f(\Delta, d) \leqq L[\alpha, \mathcal{C}_\Delta{}^*] \leqq L[\alpha, \mathcal{C}_\Delta{}^+] \leqq f(\Delta, d) + \alpha.$$

PROOF: (a) Let $\mathcal{C}$ be any class of sequences $\{C_n\}$ of functions C_n as defined by (1.1), and for every integer k define

$$c_k(\alpha) = \sup_x \{x: \inf_{n \geqq k} [d_n{}^+(x) - d_n(\alpha)] \geqq 0\}.$$

Then if $n \geqq k$ we have $d_n(\alpha) \leqq d_n{}^+(c_k)$, so that $Q_n(d_n(\alpha); C_n, C_n) \leqq Q_n(d_n{}^+(c_k); C_n, C_n) \leqq Q_n(d_n{}^+(c_k); -\infty, C_n)$. Hence, using the definition of L, we have immediately that $L[\alpha, \mathcal{C}] \geqq L^+[c_k(\alpha), \mathcal{C}]$ for every k. As $k \to \infty$, $c_k(\alpha) \to \exp(-2d^2)$ and hence $L[\alpha, \mathcal{C}] \geqq L^+[\exp(-2d^2), \mathcal{C}]$. But from (1.2) we see that $\exp(-2d^2) \geqq \frac{1}{2}\alpha$, and hence certainly $L^+[\exp(-2d^2, \mathcal{C}] \geqq L^+[\frac{1}{2}\alpha, \mathcal{C}]$.

(b) Let $\{C_n\}$ be any member of $\mathcal{C}_\Delta(\tau)$. Then for any $\beta > 0$ write $I = I(\tau, \beta)$ and define

$$A_n(t) = -\sup_{0<t<1} |C_n(t)|, \qquad 0 < t < 1$$

and

$$B_n(t) = -\Delta, \qquad t \in I,$$
$$= \sup_{0<t<1} |C_n(t)|, \qquad t \notin I.$$

Clearly $A_n(t) \leqq C_n(t)$ for $0 < t < 1$, and $C_n(t) \leqq B_n(t)$ for $t \notin I$. Hence,

$$Q_n(d_n; C_n, C_n) - Q_n(d_n; A_n, B_n)$$
$$\leqq P\{Z_n(t) \leqq C_n(t) + d_n,\ t \in I; \text{ but } Z_n(t) > B_n(t) + d_n \text{ for some } t \in I\}$$
$$\leqq P\{\inf_{t\in I} Z_n(t) \leqq \inf_{t\in I} C_n(t) + d_n,\ \sup_{t\in I} Z_n(t) \geqq \sup_{t\in I} B_n(t) + d_n\}$$
$$= P\{\inf_{t\in I} Z_n(t) \leqq \inf_{t\in I} C_n(t) + d_n \leqq \sup_{t\in I} Z_n(t) + \Delta + \inf_{t\in I} C_n(t)\}$$
$$\leqq \xi R(\tau)[\beta^{\frac{1}{2}} + \max\{0, \Delta + \inf_{t\in I} C_n(t)\} + n^{-\frac{1}{2}}]$$

by Lemma 3.4. As $n \to \infty$, $\{A_n\}$ and $\{B_n\}$ converge in the sense of Theorem 3.6 to

$$A(t) = -\Delta, \qquad 0 < t < 1,$$

and

$$B^{(\beta)}(t) = -\Delta, \qquad t \in I,$$
$$= \Delta, \qquad t \notin I$$

with T null, and hence

$$\limsup_{n\to\infty} Q_n(d_n; C_n, C_n) \leqq Q(d; -\Delta, B^{(\beta)})$$
$$+ \xi R(\tau)[\beta^{\frac{1}{2}} + \max\{0, \Delta + \limsup_{n\to\infty} \inf_{t\in I} C_n(t)\}].$$

This being true for every $\beta > 0$, we may let $\beta \to 0$ and thus obtain

$$\limsup_{n\to\infty} Q_n(d_n ; C_n , C_n) \leqq Q(d; -\Delta, B'),$$

since $\limsup_{\beta\to 0} \limsup_{n\to\infty} \inf_{t\epsilon I} C_n(t) = -\Delta$ by the definition of $\mathcal{C}_\Delta(\tau)$. Then since this result holds for every $\{C_n\} \varepsilon \mathcal{C}_\Delta(\tau)$ we have immediately the second inequality of (b) by the definition of L. But

$$Q(d; -\Delta, B') \leqq P\{-\Delta - d \leqq Z(\tau) \leqq -\Delta + d\}$$
$$= \Phi((-\Delta + d)/[\tau(1 - \tau)]^{\frac{1}{2}}) - \Phi((-\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}})$$

which is the first inequality.

Now since $\mathcal{C}_\Delta(\tau) \supset \mathcal{C}_\Delta{}^+(\tau)$ it is clear that $L[\alpha, \mathcal{C}_\Delta(\tau)] \leqq L[\alpha, \mathcal{C}_\Delta{}^+(\tau)]$, which is the third of the required inequalities.

But consider $C_n(t) = B_n(t)$ as defined by (4.2). We have $\{C_n\} \varepsilon \mathcal{C}_\Delta{}^+(\tau)$, and $\{C_n\}$ converges in the sense of Theorem 3.6 with $A(t) = 0$, $B(t) = B''(t)$, and $T = \{\tau\}$. Hence $\lim_{n\to\infty} Q_n(d_n ; C_n , C_n) = Q(d; 0, B'')$ and the fourth inequality follows immediately. But

$$1 - Q(d; 0, B'') = P\{Z(\tau) > -\Delta + d \text{ or } |Z(t)| > d \text{ for some } t, 0 < t < 1\}$$
$$\leqq P\{Z(\tau) > -\Delta + d\} + P\{\sup_{0<t<1} |Z(t)| > d\}$$
$$= \Phi((\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) + \alpha$$

which is the last of the required inequalities.

(Exact values for the two probabilities indicated here may be obtained in a straightforward manner by the method used in the proof of Lemma 2.1; however the resulting expressions are formidable and probably of little interest.)

(c) From (b) and the definitions of $\mathcal{C}_\Delta{}^*$ and $\mathcal{C}_\Delta{}^+$ it is clear that

$$\inf_{0<\tau<1} \{\Phi((\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) + \Phi((-\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}})\}$$
$$\leqq L[\alpha, \mathcal{C}_\Delta{}^*] \leqq L[\alpha, \mathcal{C}_\Delta{}^+] \leqq \inf_{0<\tau<1} \{\Phi((\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) + \alpha\}$$

and hence

$$\inf_{0<\tau<1} \Phi((\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) \leqq L[\alpha, \mathcal{C}_\Delta{}^*] \leqq L[\alpha, \mathcal{C}_\Delta{}^+]$$
$$\leqq \inf_{0<\tau<1} \Phi((\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) + \alpha;$$

and the indicated infimum is precisely $f(\Delta, d)$, obtained by letting $\tau \to 0$ if $\Delta < d(\alpha)$ and by taking $\tau = \frac{1}{2}$ if $\Delta \geqq d(\alpha)$.

We remark that if $\Delta \geqq d(\alpha) + \frac{1}{2}$ then the entire sum $\{\Phi((\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}}) + \Phi((-\Delta - d)/[\tau(1 - \tau)]^{\frac{1}{2}})\}$ is minimized by taking $\tau = \frac{1}{2}$, thus yielding the slightly better lower bound $\{\Phi(2\Delta - 2d) + \Phi(-2\Delta - 2d)\}$. It is of course possible to obtain an even better bound by using the exact probability expression for $Q(d; -\Delta, B')$ of part (c), but we have not been able to produce any simple result in this manner.

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