MOMENTS OF RANDOMLY STOPPED SUMS

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1. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space, let $x_1, x_2, \cdots$ be a sequence of random variables on Ω, and let $\mathfrak{F}_n$ be the σ-algebra generated by $x_1, \cdots, x_n$, with $\mathfrak{F}_0 = (\phi, \Omega)$. A *stopping variable* (of the sequence $x_1, x_2, \cdots$) is a random variable t on Ω with positive integer values such that the event $[t = n] \varepsilon \mathfrak{F}_n$ for every $n \geqq 1$. Let $S_n = \sum_1^n x_i$; then $S_t = S_{t(\omega)}(\omega) = \sum_1^t x_i$ is a randomly stopped sum. *We shall always assume that*

$$(1) \qquad E|x_n| < \infty, \qquad E(x_{n+1} \mid \mathfrak{F}_n) = 0, \qquad (n \geqq 1).$$

The moments of S_t have been investigated since the advent of Sequential Analysis, beginning with Wald [9], whose theorem states that for *independent, identically distributed* (iid) x_i *with* $Ex_i = 0$, $Et < \infty$ implies that $ES_t = 0$. For higher moments of S_t, the known results [1, 3, 4, 5, 10] are not entirely satisfactory. We shall obtain theorems for ES_t^r $(r = 2, 3, 4)$; the case $r = 2$ is of special interest in applications. For iid x_i with $Ex_i = 0$ and $Ex_i^2 = \sigma^2 < \infty$, we shall show that $Et < \infty$ implies $ES_t^2 = \sigma^2 Et$.

2. The second moment. It follows from assumption (1) that $(S_n, \mathfrak{F}_n; n \geqq 1)$ is a *martingale*; i.e., that

$$(2) \qquad E|S_n| < \infty, \qquad E(S_{n+1} \mid \mathfrak{F}_n) = S_n \qquad (n \geqq 1).$$

The following well-known fact ([3], p. 302) will be stated as

Lemma 1. *Let $(S_n, \mathfrak{F}_n; n \geqq 1)$ be a martingale and let t be any stopping variable such that*

$$(3) \qquad E|S_t| < \infty, \qquad \liminf \int_{[t>n]} |S_n| = 0;$$

then

$$(4) \qquad E(S_t \mid \mathfrak{F}_n) = S_n \quad if \quad t \geqq n \qquad (n \geqq 1),$$

and hence $ES_t = ES_1$.

Lemma 2. *If $E\sum_1^t |x_i| < \infty$, then (3) holds.*

Proof. $|S_t| \leqq \sum_1^t |x_i|$, so that $E|S_t| < \infty$, and

$$\lim \int_{[t>n]} |S_n| \leqq \lim \int_{[t>n]} \sum_1^t |x_i| = 0.$$

In the remainder of this section we shall suppose, in addition to (1) that

$$(5) \qquad Ex_n^2 < \infty \qquad (n \geqq 1)$$

and we define for $n \geqq 1$

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(6) $$Z_n = S_n^2 - \sum_1^n x_i^2.$$

The sequence $(Z_n, \mathfrak{F}_n; n \geqq 1)$ is also a martingale, with $EZ_1 = 0$.

For any stopping variable t, let $t(n) = \min(n, t)$; then Lemma 1 applies to Z_n and $t(n)$, so that $EZ_{t(n)} = 0$, and hence

(7) $$ES_{t(n)}^2 = E\sum_1^{t(n)} x_i^2.$$

Letting $n \to \infty$ we have $S_{t(n)}^2 \to S_t^2$ and $\sum_1^{t(n)} x_i^2 \uparrow \sum_1^t x_i^2$. Hence, by Fatou's lemma and (7),

(8) $$ES_t^2 \leqq \lim ES_{t(n)}^2 = \lim E\sum_1^{t(n)} x_i^2 = E\sum_1^t x_i^2.$$

The question now arises under what circumstances equality holds in (8). (By Lemma 1 this will be the case if (3) holds with S replaced by Z, but, as we shall see, this requirement is unnecessarily stringent.) According to (8), *we need only consider the case in which* $ES_t^2 < \infty$, *and it will suffice to prove that*

(9) $$ES_t^2 \geqq ES_{t(n)}^2 \qquad (n \geqq 1).$$

LEMMA 3. *If*

(10) $$\liminf \int_{[t>n]} |S_n| = 0,$$

then $ES_t^2 = E\sum_1^t x_i^2$.

PROOF. We may suppose that $ES_t^2 < \infty$ whence, by (10) and Lemma 1, (4) holds. Hence

$$\begin{aligned} ES_t^2 &= \int_{[t\leqq n]} S_t^2 + \int_{[t>n]} (S_n + (S_t - S_n))^2 \\ &\geqq \int_{[t\leqq n]} S_t^2 + \int_{[t>n]} S_n^2 + 2\int_{[t>n]} S_n E(S_t - S_n \mid \mathfrak{F}_n) = ES_{t(n)}^2. \end{aligned}$$

LEMMA 4. *If*

(11) $$\liminf \int_{[t>n]} S_n^2 < \infty,$$

then (10) *holds.*

PROOF. Suppose (10) does not hold; then $\liminf \int_{[t>n]} |S_n| = \epsilon > 0$. Hence for any constant $0 < a < \infty$,

$$\liminf \int_{[t>n]} S_n^2 \geqq a \liminf \int_{[t>n, |S_n|>a]} |S_n| = a\epsilon,$$

which contradicts (11), since a may be arbitrarily large.

LEMMA 5. *If* $E\sum_1^t x_i^2 < \infty$, *then* (11) *holds.*

PROOF. Setting $S_0 = 0$ we have

$$\begin{aligned} \int_{[t>n]} S_n^2 &= \sum_{i=1}^n \left(\int_{[t>i]} S_i^2 - \int_{[t>i-1]} S_{i-1}^2\right) \\ &\leqq \sum_{i=1}^n \int_{[t\geqq i]} (S_i^2 - S_{i-1}^2) \leqq \sum_1^\infty \int_{[t\geqq i]} x_i^2 = E\sum_1^t x_i^2 < \infty. \end{aligned}$$

From Lemmas 1–5 we have

THEOREM 1. *Let* $(S_n, \mathfrak{F}_n; n \geqq 1)$ *be a martingale with* $ES_n^2 < \infty$ *and let* t *be any stopping variable. Set* $x_1 = S_1$, $x_{n+1} = S_{n+1} - S_n$. *Then*

(12) $$ES_t^2 \leqq E\sum_1^t x_i^2.$$

If any one of the four conditions

(13) $$\liminf \int_{[t>n]} |S_n| = 0, \qquad \liminf \int_{[t>n]} S_n^2 < \infty,$$
$$E\sum_1^t |x_i| < \infty, \qquad E\sum_1^t x_i^2 < \infty$$

holds, then

(14) $$ES_t^2 = E\sum_1^t x_i^2.$$

If either $E\sum_1^t |x_i| < \infty$ or $E\sum_1^t x_i^2 < \infty$, *then* (3) *and* (4) *hold.*

Theorem 1 generalizes (a) and (b) of Theorem II of [1]. In order to apply it, we first verify

LEMMA 6. *For any stopping variable t and any* $r > 0$,

$$E\sum_1^t |x_i|^r = E\sum_1^t E(|x_i|^r|\mathfrak{F}_{i-1}).$$

PROOF.

$$E\sum_1^t |x_i|^r = \sum_{j=1}^\infty \int_{[t=j]} \sum_{i=1}^j |x_i|^r = \sum_{i=1}^\infty \int_{[t\geqq i]} |x_i|^r$$
$$= \sum_{i=1}^\infty \int_{[t\geqq i]} E(|x_i|^r \mid \mathfrak{F}_{i-1}) = E\sum_1^t E(|x_i|^r \mid \mathfrak{F}_{i-1}).$$

For independent x_n, we have from Theorem 1 and Lemmas 1 and 6

THEOREM 2. *Let* $x_1, x_2, \cdots$ *be independent with* $Ex_n = 0$, $E|x_n| = a_n$, $Ex_n^2 = \sigma_n^2 < \infty\,(n \geqq 1)$ *and let* $S_n = \sum_1^n x_i$. *Then if t is a stopping variable, either of the two relations*

(15) $$E\sum_1^t a_i < \infty, \qquad E\sum_1^t \sigma_i^2 < \infty$$

implies that $ES_t = 0$ *and*

(16) $$ES_t^2 = E\sum_1^t x_i^2 = E\sum_1^t \sigma_i^2.$$

If $\sigma_n^2 = \sigma^2 < \infty$, *then* $Et < \infty$ *implies*

(17) $$ES_t^2 = E\sum_1^t x_i^2 = \sigma^2 Et.$$

Some stronger sufficient conditions for (17) have been given in ([10], [1], [5], [3] (p. 351), [4]).

COROLLARY 1. *Let* $x_1, x_2, \cdots$ *be independent with* $Ex_n = 0$, $Ex_n^2 = 1$, *and define* t^*(resp. t_*) $= 1\text{st }n \geqq 1$ *such that* $|S_n| > n^{\frac{1}{2}}$(resp. $<$)($= \infty$ *otherwise*). *Then* $Et^* = Et_* = \infty$.

PROOF. If $Et^* < \infty$, then t^* is a genuine stopping variable, i.e., $P(t^* < \infty) = 1$, and by the definition of t^* and (17),

$$Et^* = ES_{t^*}^2 > Et^*,$$

a contradiction; similarly for t_*.

We note that both t^* and t_* are genuine stopping variables if the x_n are, in addition, identically distributed.

The example $P[x_n = 1] = P[x_n = -1] = \frac{1}{2}$ shows that the $>$ ($<$) cannot be replaced by $\geqq$ ($\leqq$), since $Ex_n = 0$, $Ex_n^2 = 1$, and $t^* = t_* = 1$. On the other hand, if t^* is redefined as the first $n > 1$ for which $|S_n| \geqq n^{\frac{1}{2}}$, Et^* is again infinite; similarly for t_*.

Corollary 1 is a generalization of Theorem 1 of [2]. The following corollary generalizes Theorem 2 of [2].

COROLLARY 2. *Let* $x_1, x_2, \cdots$ *be independent with* $Ex_n = 0$, $Ex_n^2 = 1$, $P[|x_n| \leqq a < \infty] = 1$. *For* $0 < c < 1$ *and* $m = 1, 2, \cdots$, *define* $t =$ *first* $n \geqq m$ *such that* $|S_n| > cn^{\frac{1}{2}}$. *Then* $Et < \infty$.

PROOF. For $k = m, m+1, \cdots$, put $t' = \min(t, k)$ and $A_k = [\omega: m < t \leqq k]$. Then t' is a stopping variable and by Theorem 2

$$kP[t > k] + \int_{[t \leqq k]} t = Et' = ES_{t'}^2 = \int_{[t>k]} S_k^2 + \int_{[t \leqq k]} S_t^2$$

or

$$kP[t > k] + \int_{A_k} t \leqq c^2 kP[t > k] + \int_{A_k} (ct^{\frac{1}{2}} + a)^2 + m.$$

Hence

$$(1 - c^2)(kP[t > k] + \int_{A_k} t) \leqq 2ac \int_{A_k} t^{\frac{1}{2}} + O(1).$$

Therefore, as $k \to \infty$, $\int_{A_k} t = O(1)$ and $P[t > k] = O(k^{-1}) = o(1)$, so that t is a genuine stopping variable and $Et < \infty$.

COROLLARY 3. *If* $x_1, x_2, \cdots$, *are* iid *with* $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $P[|x_n| \leqq a < \infty] = 1$, *and if* $ES_t^2 < \infty$ *for a stopping variable* t, *then* $Et < \infty$ *if and only if*

$$\liminf nP[t > n] = 0. \tag{18}$$

PROOF. The "only if" part is obvious. Now suppose (18) holds. Then since $\int_{[t>n]} |S_n| \leqq anP[t > n]$, the first condition of (13) holds and hence $\sigma^2 Et = ES_t^2 < \infty$, so that $Et < \infty$ if $\sigma^2 > 0$. (If $\sigma^2 = 0$, then $P[x_n = 0] = 1$ and hence t is equal a.e. to a fixed positive integer, so $Et < \infty$ in this case too.)

Applied to the case $P[x_i = 1] = P[x_i = -1] = \frac{1}{2}$, with $t =$ first $n \geqq 1$ such that $S_t = 1$, we have by Wald's theorem $Et = \infty$, but by Corollary 3 the stronger result $\liminf nP[t > n] > 0$.

COROLLARY 4. *Let* $(x_n, n \geqq 1)$ *satisfy* $E(x_{n+1} \mid \mathfrak{F}_n) = 0$ *and let* $E(x_{n+1}^2 \mid \mathfrak{F}_n) = \sigma_{n+1}^2 < \infty$ *be constant for* $n \geqq 0$. *Then for* $\epsilon > 0$,

$$P[\max_{n \leqq m} |S_n| \geqq \epsilon] \leqq \epsilon^{-2} \sum_1^m \sigma_n^2.$$

If moreover $\sup_{n \geqq 1} |x_n| = z$ *with* $Ez < \infty$, *then*

$$P[\max_{n \leqq m} |S_n| \geqq \epsilon] \geqq 1 - [E(\epsilon + z)^2 / \textstyle\sum_1^m \sigma_n^2]. \tag{19}$$

PROOF. Define $t =$ first $n \geqq 1$ such that $|S_n| \geqq \epsilon$. Then $t' = \min(t, m)$ is a bounded stopping variable. Hence, by (14) and Lemma 6,

$$\epsilon^2 P[\max_{n \leqq m} |S_n| \geqq \epsilon] = \epsilon^2 P[t \leqq m] \leqq ES_{t'}^2 = E \sum_1^{t'} \sigma_n^2 \leqq \sum_1^m \sigma_n^2.$$

If $Ez < \infty$, then

$$E(\epsilon + z)^2 \geqq ES_{t'}^2 = E \sum_1^{t'} \sigma_n^2 \geqq \int_{[t \geqq m]} \sum_1^m \sigma_j^2 = (\textstyle\sum_1^m \sigma_j^2) P[t \geqq m]$$

and (19) holds.

The first part of Corollary 4 is a special case of submartingale inequalities ([6], p. 391), and the second part generalizes slightly one of the Kolmogorov inequalities ([6], p. 235) which requires that z be constant.

3. The fourth moment. The analysis in the case of the fourth moment of S_t is somewhat easier than that of the third moment and consequently is presented first. In this section Ex_n^4 will be supposed finite and $Ex_1 = 0$. Define for $r = 1, 2, 3, 4$, and $n = 1, 2, \cdots$

$$(20)\qquad \begin{aligned} u_{r,n} &= E(x_n^r \mid \mathfrak{F}_{n-1}), & U_{r,n} &= \textstyle\sum_1^n u_{r,j}, \\ v_{r,n} &= E(|x_n|^r \mid \mathfrak{F}_{n-1}), & V_{r,n} &= \textstyle\sum_1^n v_{r,j}, \\ T_{r,n} &= \textstyle\sum_1^n |x_j|^r, & T_{1,n} &= T_n . \end{aligned}$$

In these terms, Lemma 6 asserts that $ET_{r,t} = EV_{r,t}$.

LEMMA 7. *If* $ES_t^2 < \infty$ *and* $\liminf \int_{[t>n]} |S_n| = 0$, *then*

$$E(S_t^2 \mid \mathfrak{F}_n) \geqq S_n^2 \quad and \quad E(|S_t| \mid \mathfrak{F}_n) \geqq |S_n| \qquad for \quad t > n.$$

PROOF. For any $A \in \mathfrak{F}_n$, by Lemma 1

$$\int_{A[t>n]} S_t^2 = \int_{A[t>n]} [S_n^2 + 2S_n(S_t - S_n) + (S_t - S_n)^2] \geqq \int_{A[t>n]} S_n^2.$$

Hence the first inequality of the lemma holds, and the second inequality follows immediately from Lemma 1 and the fact that $E(|S_t| \mid \mathfrak{F}_n) \geqq |E(S_t \mid \mathfrak{F}_n)|$.

THEOREM 3. *If t is a stopping variable such that* $E[t \sum_1^t E(x_j^4 \mid \mathfrak{F}_{j-1})] < \infty$, *then* $ES_t^4 < \infty$ *and*

$$(21)\qquad ES_t^4 = EU_{4,t} + 4ES_tU_{3,t} + 6ES_t^2U_{2,t} - 6E \textstyle\sum_1^t u_{2,j}U_{2,j} .$$

PROOF. Set $Y_n = S_n^4 - 6S_n^2U_{2,n} - 4S_nU_{3,n} - U_{4,n} + 6 \sum_{j=1}^n u_{2,j} U_{2,j}$ and $t' = \min (t, k)$. Since $\{Y_n, \mathfrak{F}_n; n \geqq 1\}$ is a martingale with $EY_1 = 0$, by Lemma 1,

$$\begin{aligned} ES_{t'}^4 &= 6ES_{t'}^2U_{2,t'} + 4ES_{t'}U_{3,t'} + EU_{4,t'} - 6E(\textstyle\sum_{j=1}^{t'} u_{2,j}U_{2,j}) \\ &\leqq 6(E^{\frac12}S_{t'}^4)(E^{\frac12}U_{2,t'}^2) + 4(E^{\frac14}S_{t'}^4)(E^{3/4}V_{3,t'}^{4/3}) + EU_{4,t'} , \end{aligned}$$

whence, if $ES_{t'}^4 > 0$,

$$(22)\qquad E^{\frac12}S_{t'}^4 \leqq 6E^{\frac12}U_{2,t'}^2 + 4(E^{3/4}V_{3,t'}^{4/3})(ES_{t'}^4)^{-\frac14} + (EU_{4,t'})(ES_{t'}^4)^{-\frac12}.$$

Now if $p > 1$, $r > 0$,

$$(23)\qquad \begin{aligned} V_{r,n} = \textstyle\sum_{j=1}^n E\{|x_j|^r \mid \mathfrak{F}_{j-1}\} &\leqq n^{(p-1)/p}(\textstyle\sum_{j=1}^n E^p\{|x_j|^r \mid \mathfrak{F}_{j-1}\})^{1/p} \\ &\leqq n^{(p-1)/p}(\textstyle\sum_{j=1}^n E\{|x_j|^{pr} \mid \mathfrak{F}_{j-1}\})^{1/p} = n^{(p-1)/p}V_{pr,n}^{1/p} \end{aligned}$$

and thus setting $p = 2$, $r = 2$ and then $p = \frac43$, $r = 3$,

$$(24)\qquad EU_{2,t}^2 = EV_{2,t}^2 \leqq EtV_{4,t} < \infty, \qquad EV_{3,t}^{4/3} \leqq Et^{\frac13}V_{4,t} < \infty.$$

Moreover, $EU_{4,t} \leqq E(tU_{4,t}) < \infty$ and $E(\sum_{j=1}^t u_{2,j}U_{2,j}) \leqq EU_{2,t}^2 < \infty$. Thus, the LHS of (22) is a bounded function of k, implying via Fatou's lemma that $ES_t^4 < \infty$.

Since $|Y_n| \leqq S_n^4 + 6S_n^2U_{2,n} + 4|S_n| V_{3,n} + U_{4,n} + 6\sum_{j=1}^n u_{2,j}U_{2,j} = Y_n'$ (say), it follows from the preceding that

$$E|Y_t| \leqq EY_t' \leqq ES_t^4 + 6(E^{\frac{1}{2}}S_t^4)(E^{\frac{1}{2}}U_{2,t}^2) + 4(E^{\frac{1}{4}}S_t^4)(E^{3/4}V_{3,t}^{4/3}) + EU_{4,t} + 6EU_{2,t}^2 < \infty.$$

From (24), $ET_{2,t} = EU_{2,t} < \infty$. Thus, (8) of Section 2 and Lemmas 4 and 5 are valid, whence by Lemma 7, $E\{S_t^2 \mid \mathfrak{F}_k\} \geqq S_k^2$ for $t > k$, $k = 1, 2, \cdots$. Consequently,

$$\int_{[t>n]} S_t^4 = \int_{[t>n]} [S_n^4 + 2S_n^2(S_t^2 - S_n^2) + (S_t^2 - S_n^2)^2 \geqq \int_{[t>n]} S_n^4 + 2\int_{[t>n]} S_n^2E\{S_t^2 - S_n^2 \mid \mathfrak{F}_n\} \geqq \int_{[t>n]} S_n^4$$

implying $\int_{[t>n]} S_n^4 = o(1)$ and concomitantly

$$\begin{aligned} \int_{[t>n]} S_n^2U_{2,n} &\leqq (\textstyle\int_{[t>n]} S_n^4)^{\frac{1}{2}}(\int_{[t>n]} U_{2,t}^2)^{\frac{1}{2}} = o(1), \\ \int_{[t>n]} |S_n| V_{3,n} &\leqq (\textstyle\int_{[t>n]} S_n^4)^{\frac{1}{4}}(\int_{[t>n]} V_{3,t}^{4/3})^{3/4} = o(1), \\ \int_{[t>n]} U_{4,n} &\leqq \textstyle\int_{[t>n]} U_{4,t} = o(1), \\ \int_{[t>n]} \textstyle\sum_{j=1}^n u_{2,j}U_{2,j} &\leqq \textstyle\int_{[t>n]} U_{2,n}^2 \leqq \int_{[t>n]} U_{2,t}^2 = o(1). \end{aligned} \tag{25}$$

Thus, $\int_{[t>n]} |Y_n| \leqq \int_{[t>n]} Y_n' = o(1)$ and by Lemma 1, $EY_t = EY_1 = 0$.

Alternative expressions for ES_t^4 are possible as indicated in

THEOREM 4. *If $E(t\sum_{j=1}^t E\{x_j^4 \mid \mathfrak{F}_{j-1}\}) < \infty$, then setting $S_0 = 0$,*

$$ES_t^4 = 6E\sum_{j=1}^t S_{j-1}^2u_{2,j} + 4E\sum_{j=1}^t S_{j-1}u_{3,j} + EU_{4,t}.$$

The proof of Theorem 4 is similar to that of Theorem 3 and will be omitted.

COROLLARY. *If $E(tU_{4,t}) < \infty$, then*

$$E(6\sum_{j=1}^t S_{j-1}^2u_{2,j} + 4\sum_{j=2}^t S_{j-1}u_{3,j}) = 6ES_t^2U_{2,t} + 4ES_tU_{3,t} - 6E(\sum_{j=1}^t u_{2,j}U_{2,j}).$$

It is intuitively clear that terms with like coefficients are equal, and indeed we have

LEMMA 8. *If $E(tU_{4,t}) < \infty$, then $ES_tU_{3,t} = E(\sum_{j=2}^t S_{j-1}u_{3,j})$ and $E(S_t^2U_{2,t}) = E(\sum_{j=2}^t S_{j-1}^2 u_{2,j}) + E(\sum_{j=1}^t u_{2,j}U_{2,j})$.*

PROOF. It suffices to verify the first of the two relationships since the second will then follow from the corollary to Theorem 4. Suppose first that

$$E(\sum_{j=1}^t |x_j| V_{r,j}) < \infty. \tag{26}$$

Then

$$\sum_{k=1}^\infty \int_{[t=k]} \sum_{j=1}^k x_jU_{r,j} = \sum_{j=1}^\infty \int_{[t\geqq j]} x_jU_{r,j} = \sum_{j=1}^\infty \int_{[t\geqq j]} E(x_j \mid \mathfrak{F}_{j-1})U_{r,j} = 0,$$

whence

$$E\left(\sum_{j=1}^{t} S_{j-1}u_{r,j}\right) = \sum_{k=1}^{\infty} \int_{[t=k]} \left[\sum_{j=2}^{k} S_{j-1}u_{r,j} + \sum_{j=1}^{k} x_j U_{r,j}\right] \tag{27}$$
$$= \sum_{k=1}^{\infty} \int_{[t=k]} S_k U_{r,k}$$
$$= ES_t U_{r,t}.$$

Thus, if $t' = \min(t, N)$, (27) holds with t replaced by t' irrespective of (26). However,

$$ES_tU_{3,t} = \sum_{k=1}^{N} \int_{[t=k]} S_kU_{3,k} + \int_{[t>N]} S_tU_{3,t} \tag{28}$$
$$= ES_{t'}U_{3,t'} - \int_{[t>N]} S_NU_{3,N} + \int_{[t>N]} S_tU_{3,t},$$

and analogously

$$E\left(\sum_{j=2}^{t} S_{j-1}u_{3,j}\right) = E\left(\sum_{j=2}^{t'} S_{j-1}u_{3,j}\right) - \int_{[t>N]} \sum_{j=2}^{N} S_{j-1}u_{3,j} + \int_{[t>N]} \sum_{j=2}^{t} S_{j-1}u_{3,j}. \tag{29}$$

Now $E|S_tU_{3,t}| \leqq EY_t' < \infty$, and employing Lemma 7,

$$E \sum_{1}^{t} |S_{j-1}u_{3,j}| = \sum_{k=1}^{\infty} \int_{[t=k]} \sum_{1}^{k} |S_{j-1}u_{3,j}| = \sum_{j=1}^{\infty} \int_{[t\geqq j]} |S_{j-1}u_{3,j}|$$
$$\leqq \sum_{1}^{\infty} \int_{[t\geqq j]} |S_tu_{3,j}| \leqq E|S_t| V_{3,t} \leqq EY_t' < \infty.$$

These facts plus (25) imply that all unwanted terms of (28) and (29) are $o(1)$ and the result follows.

Identities and inequalities analogous to (27) abound and several of these will be catalogued as

LEMMA 9. *$E(\sum_{n=1}^{t} S_n^2) \leqq EtS_t^2$ under the conditions of Lemma* 7.

$$E\left(\sum_{n=1}^{t} S_n\right) = EtS_t \quad \text{if} \quad EtT_t < \infty.$$
$$E\left(\sum_{n=1}^{t} T_n\right) \leqq EtT_t \quad \text{if} \quad EtT_t < \infty.$$

PROOF.

$$E \sum_{n=1}^{t} S_n^2 = \sum_{k=1}^{\infty} \int_{[t=k]} \sum_{n=1}^{k} S_n^2$$
$$= \sum_{n=1}^{\infty} \int_{[t\geqq n]} S_n^2 \leqq \sum_{n=1}^{\infty} \int_{[t\geqq n]} E(S_t^2 \mid \mathfrak{F}_n)$$
$$= \sum_{n=1}^{\infty} \int_{[t\geqq n]} S_t^2 = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \int_{[t=k]} S_t^2$$
$$= \sum_{k=1}^{\infty} k \int_{[t=k]} S_t^2 = EtS_t^2$$

employing Lemma 7. Similarly,

$$E\left(\sum_{n=1}^{t} T_n\right) = \sum_{n=1}^{\infty} \int_{[t\geqq n]} T_n \leqq \sum_{n=1}^{\infty} \int_{[t\geqq n]} T_t = EtT_t.$$

Finally,

$$E\left(\sum_{n=1}^{t} S_n\right) = \sum_{n=1}^{\infty} \int_{[t\geqq n]} S_n = \sum_{n=1}^{\infty} \int_{[t\geqq n]} E(S_t \mid \mathfrak{F}_n) = \sum_{n=1}^{\infty} \int_{[t\geqq n]} S_t$$
$$= EtS_t$$

in view of Lemmas 1 and 2 and the validity of interchanging the order of summation and integration.

4. The third moment. In this section $E(|x_n|^3)$ will be supposed finite and $Ex_1 = 0$. Define

$$
\begin{aligned}
Y_n &= S_n^3 - 3S_n U_{2,n} - U_{3,n}, \\
W_n &= S_n^3 - 3 \textstyle\sum_{j=1}^n S_{j-1} u_{2,j} - U_{3,n}, \qquad (30)\\
Z_n &= S_n^3 - 3 \textstyle\sum_{j=1}^n S_j u_{2,j} - U_{3,n}.
\end{aligned}
$$

It is readily checked that $(Y_n, \mathfrak{F}_n; n \geqq 1)$, $(W_n, \mathfrak{F}_n; n > 1)$, $(Z_n, \mathfrak{F}_n; n > 1)$ are all martingales and that $EY_1 = EW_1 = EZ_1 = 0$.

THEOREM 5. *If* $EV_{3,t} < \infty$ *and* $EV_{1,t}^3 < \infty$, *or equivalently if* $ET_t^3 < \infty$, *then* $E|S_t|^3 < \infty$ *and* $ES_t^3 = 3E(\sum_{j=1}^t S_{j-1} u_{2,j}) + EU_{3,t}$.

PROOF. Suppose that $EV_{3,t} < \infty$, $EV_{1,t}^3 < \infty$. (Their equivalence with $ET_t^3 < \infty$ will be deferred to Lemma 10). Then

$$
\begin{aligned}
E|S_t|^3 &= \textstyle\sum_{k=1}^\infty \int_{[t=k]} \sum_{n=1}^k (|S_n|^3 - |S_{n-1}|^3) \leqq \sum_{k=1}^\infty \sum_{n=1}^k \int_{[t=k]} (|x_n|^3 \\
&\qquad + 3|S_{n-1}| x_n^2 + 3S_{n-1}^2 |x_n|) \qquad (31)\\
&\leqq 6 \textstyle\sum_{k=1}^\infty \sum_{n=1}^k \int_{[t=k]} (|x_n|^3 + S_{n-1}^2 |x_n|) \\
&= 6[E(\textstyle\sum_{n=1}^t |x_n|^3) + E(\sum_{n=1}^t S_{n-1}^2 |x_n|)].
\end{aligned}
$$

By Lemma 6,

$$E(\textstyle\sum_{n=1}^t |x_n|^3) = EV_{3,t} < \infty. \qquad (32)$$

On the other hand, $ES_t^2 \leqq ET_t^2 \leqq 1 + ET_t^3 < \infty$ and

$$\int_{[t>k]} |S_k| \leqq \int_{[t>k]} T_k \leqq \int_{[t>k]} T_t \leqq \int_{[t>k]} (1 + T_t^3) = o(1)$$

in view of the asserted equivalence. Thus, Lemma 7 holds, whence

$$
\begin{aligned}
&E(\textstyle\sum_{n=1}^t S_{n-1}^2 |x_n|) \\
&\quad = \textstyle\sum_{k=1}^\infty \sum_{n=1}^k \int_{[t=k]} S_{n-1}^2 |x_n| = \sum_{n=1}^\infty \int_{[t\geqq n]} S_{n-1}^2 v_{1,n} \qquad (33)\\
&\quad \leqq \textstyle\sum_{n=1}^\infty \int_{[t\geqq n]} E(S_t^2 \mid \mathfrak{F}_{n-1}) v_{1,n} = \sum_{n=1}^\infty \int_{[t\geqq n]} S_t^2 v_{1,n} \\
&\quad = \textstyle\sum_{k=1}^\infty \sum_{n=1}^k \int_{[t=k]} S_t^2 v_{1,n} = ES_t^2 V_{1,t} \leqq (E^{2/3} |S_t|^3)(E^{\frac{1}{3}} V_{1,t}^3).
\end{aligned}
$$

Replace t by $t' = \min(t, k)$ in (31). Then from (32) and (33),

$$E|S_{t'}|^3 \leqq 6EV_{3,t'} + 6(E^{2/3} |S_{t'}|^3)(E^{\frac{1}{3}} V_{1,t'}^3) = O(1) + O(1) E^{2/3} |S_{t'}|^3$$

whence, by Fatou's lemma,

$$E|S_t|^3 < \infty. \qquad (34)$$

Next, (34) implies that the expectation in the LHS of (33) is finite whence

$$
\begin{aligned}
E(\textstyle\sum_{n=1}^t |S_{n-1}| u_{2,n}) &= \textstyle\sum_{n=1}^\infty \int_{[t\geqq n]} |S_{n-1}| x_n^2 = E(\sum_{n=1}^t |S_{n-1}| x_n^2) \qquad (35)\\
&\leqq E[\textstyle\sum_{n=1}^t (|x_n|^3 + |S_{n-1}|^2 |x_n|)] < \infty.
\end{aligned}
$$

Combining (34) and (35), $E|W_t| < \infty$. Since, paralleling (31),

$$\int_{[t>k]} |S_k|^3 \leqq 6 \int_{[t>k]} \sum_{n=1}^{k} (|x_n|^3 + S_{n-1}^2 |x_n|) = o(1),$$

$\int_{[t>k]} |W_k| = o(1)$ and the theorem follows from Lemma 1.

LEMMA 10. *$EV_{3,t} < \infty$ and $EV_{1,t}^3 < \infty$ if and only if $ET_t^3 < \infty$.*

PROOF. Suppose $EV_{3,t} < \infty$ and $EV_{1,t}^3 < \infty$. The argument of (31) with T_t replacing S_t yields

$$ET_t^3 \leqq 6 \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t=k]} (|x_n|^3 + T_{n-1}^2 |x_n|).$$

The inequality of (33) also obtains with T replacing S in view of the fact that $T_t \geqq T_{n-1}$ on the set $[t \geqq n]$. Thus, analogously, $ET_{t'}^3 \leqq O(1) + O(1)E^{2/3}T_{t'}^3$, implying $ET_t^3 < \infty$.

Conversely, if $ET_t^3 < \infty$, clearly $EV_{3,t} = ET_{3,t} \leqq ET_t^3 < \infty$. Moreover,

$$\begin{aligned} EV_{1,t'}^3 &= \sum_{j=1}^{\infty} \int_{[t'=j]} \sum_{n=1}^{j} (V_{1,n}^3 - V_{1,n-1}^3) \\ &\leqq \sum_{j=1}^{\infty} \sum_{n=1}^{j} \int_{[t'=j]} (v_{1,n}^3 + 3V_{1,n-1}^2 v_{1,n} + 3V_{1,n-1} v_{1,n}^2) \\ &\leqq O(1) + 6 \sum_{j=1}^{\infty} \sum_{n=1}^{j} \int_{[t'=j]} V_{1,n-1}^2 v_{1,n} \\ &= O(1) + 6 \sum_{n=1}^{\infty} \int_{[t' \geqq n]} |x_n| V_{1,n-1}^2 \\ &\leqq O(1) + 6 \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} \int_{[t'=j]} |x_n| V_{1,t'}^2 \\ &\leqq O(1) + 6\, ET_{t'} V_{1,t'}^2 \\ &\leqq O(1) + O(1) E^{2/3} V_{1,t'}^3, \end{aligned}$$

which implies, as earlier, that $EV_{1,t}^3 < \infty$ and completes the proof.

THEOREM 6. *If $ET_t^3 < \infty$ and $Et^{\frac{1}{2}}V_{3,t} < \infty$, $ES_t^3 = 3ES_tU_{2,t} + EU_{3,t} < \infty$.*

PROOF. As in Theorem 5, after setting $p = \frac{3}{2}$, $r = 2$ in (23) of Section 3 to obtain

$$E|S_tU_{2,t}| \leqq (E^{\frac{1}{3}}|S_t|^3)(E^{2/3}U_{2,t}^{3/2}) \leqq (E^{\frac{1}{3}}|S_t|^3)(E^{2/3}t^{\frac{1}{2}}V_{3,t}).$$

COROLLARY 1. *Under the conditions of Theorem* 6, $E(\sum_{j=1}^{t} x_j u_{2,j}) = 0$.

PROOF. Analogously, $EZ_t = 0$, whence $E(W_t - Z_t) = 0$.

COROLLARY 2. *Under the conditions of Theorem* 6, $ES_tU_{2,t} = E(\sum_{j=1}^{t} S_{j-1}u_{2,j})$.

The single requirement $ET_t^3 < \infty$, although equivalent to the two conditions of Theorem 5, is difficult to check. The following single condition is easily seen to imply all those of Theorems 5 and 6:

$$E(t^2V_{3,t}) < \infty, \tag{36}$$

and in addition yields

$$\begin{aligned} ET_t^3 = {} & 3ET_t^2V_{1,t} + 3ET_t(V_{2,t} - 2\sum_{j=1}^{t} V_{1,j}v_{1,j}) + EV_{3,t} \\ & - 3E(\sum_{j=1}^{t} V_{1,j}v_{2,j}) - 3E(\sum_{j=1}^{t} V_{2,j}v_{1,j}) + 6E(\sum_{j=1}^{t} v_{1,j} \sum_{i=1}^{j} V_{1,i}v_{1,i}). \end{aligned}$$

5. Sums of independent random variables. In this section, the random variables $x_1, x_2, \cdots$ will be supposed independent. If $Ex_n = 0$, all prior theorems are, of course, applicable but may be reformulated in especially simple terms with conditions that are susceptible of immediate verification. For example, from Theorems 3 and 6, we obtain:

THEOREM 7. *If $x_1, x_2, \cdots$ are independent with $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $Ex_n^3 = \gamma$, $Ex_n^4 = \beta < \infty$ and t is a stopping rule with $Et^2 < \infty$, then $ES_t^4 < \infty$ and*

$$ES_t^4 = 6\sigma^2 EtS_t^2 + 4\gamma EtS_t + \beta Et - 3\sigma^4 Et(t+1).$$

THEOREM 8. *If $x_1, x_2, \cdots$ are independent with $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $Ex_n^3 = \gamma$, $E|x_n|^3 \leqq C < \infty$, and if t is a stopping variable with $Et^3 < \infty$, then $ES_t^3 = \gamma Et + 3\sigma^2 EtS_t < \infty$.*

PROOF. According to Theorem 6 and Lemma 10, it suffices to verify that

$$EV_{3,t} \leqq E(t^{\frac{1}{2}} V_{3,t}) \leqq CEt^{3/2} < \infty,$$

$$EV_{1,t}^3 \leqq E[t(1+C)]^3 < \infty.$$

In the final theorem, the requirement of Theorem 8 that $Et^3 < \infty$ will be relaxed at the expense of increasing the moment assumptions on x_n.

THEOREM 9. *If $x_1, x_2, \cdots$ are independent with $Ex_n = 0$, $Ex_n^2 = \sigma^2$, $Ex_n^3 = \gamma$, $Ex_n^4 \leqq C < \infty$, and if t is a stopping variable with $Et^2 < \infty$, then $ES_t^3 = \gamma Et + 3\sigma^2 EtS_t$.*

PROOF. Here, the martingale Y_n of (30) simplifies to $Y_n = S_n^3 - 3\sigma^2 nS_n - n\gamma$. The theorem will follow from Lemmas 1 and 2 once it is established that $E\sum_1^t |Y_{n+1} - Y_n| = E\sum_1^t E(|Y_{n+1} - Y_n| \mid \mathfrak{F}_n) < \infty$. Now if B and D are finite constants (not necessarily the same in each appearance),

$$E(|S_{n+1}^3 - S_n^3| \mid \mathfrak{F}_n) \leqq 6E(|x_{n+1}|^3 + S_n^2 |x_{n+1}| \mid \mathfrak{F}_n) \leqq BS_n^2 + D,$$

$$E(|(n+1)S_{n+1} - nS_n| \mid \mathfrak{F}_n) = E(|S_n + (n+1)x_{n+1}| \mid \mathfrak{F}_n) \leqq S_n^2 + nD,$$

whence

$$E(|Y_{n+1} - Y_n| \mid \mathfrak{F}_n) \leqq BS_n^2 + nD.$$

Next, Lemma 9 is applicable below since (17) insures $ES_t^2 < \infty$ while Lemmas 6 and 2 guarantee (10). Consequently,

$$\begin{aligned} E\sum_1^t E(|Y_{n+1} - Y_n| \mid \mathfrak{F}_n) &\leqq B\cdot E(\textstyle\sum_1^t S_n^2) + D\cdot Et(t+1) \\ &\leqq B\cdot EtS_t^2 + D\cdot Et(t+1) \\ &\leqq B\cdot(E^{\frac{1}{2}}t^2)(E^{\frac{1}{2}}S_t^4) + D\cdot Et(t+1) < \infty. \end{aligned}$$

REFERENCES

[1] BLACKWELL, D. and GIRSHICK, M. A. (1947). A lower bound for the variance of some unbiased sequential estimates. *Ann. Math. Statist.* **18** 277–280.

[2] BLACKWELL, D. and FRIEDMAN, D. (1964). A remark on the coin tossing game. *Ann. Math. Satist.* **35** 1345–1347.

[3] DOOB, J. L. (1953). *Stochastic Processes.* Wiley, New York.

[4] JOHNSON, N. L. (1959). A proof of Wald's theorem on cumulative sums. *Ann. Math. Statist.* **30** 1245–1247.

[5] KOLMOGOROV, A. N. and PROHOROV, Y. (1949). On sums of a random number of random terms. *Uspehi Mat. Nauk* **4** 168–172 (In Russian).

[6] Loève, M. (1960). *Probability Theory.* (2nd ed.). Van Nostrand, Princeton.

[7] SEITZ, J. and WINKELBAUER, K. (1953). Remarks concerning a paper of Kolmogorov and Prohorov. *Czechoslovak. Math. J.* **3** 89–91. (In Russian, with English summary.)

[8] WINKELBAUER, K. (1953). Moments of cumulative sums of random variables. *Czechoslovak. Math. J.* **3** 93–108.

[9] WALD, A. (1944). On cumulative sums of random variables. *Ann. Math. Statist.* **15** 283–296.

[10] WOLFOWITZ, J. (1947). The efficiency of sequential estimates. *Ann. Math. Statist.* **18** 215–230.