A SIMPLE PROBABILISTIC PROOF OF THE DISCRETE GENERALIZED RENEWAL THEOREM

By Sidney C. Port

The RAND Corporation

Let $\{X_n\}$, $n > 0$ be a sequence of independent, identically distributed, integer-valued random variables with partial sums $S_n = X_1 + X_2 + \cdots + X_n$ ($S_0 = 0$). Throughout this discussion we will always assume that the distribution of X_1 is aperiodic (i.e., that the minimal additive subgroup generated by the points of increase of the distribution of X_1 is the group of all integers). Consider the Markov chain on the integers with transition probability $P(x, y) = P(X_1 = y - x)$. Following the terminology of [7], we call such a Markov chain a random walk. Assume $EX_1 > 0$ and also the possibility that $EX_1 = +\infty$. (More precisely we are assuming that $E[\max(0, X_1)] > E[-\min(0, X_1)]$ and $E[-\min(0, X_1)] < \infty$. However, we allow the possibility that $E\max(0, X_1) = \infty$.) Then the strong law of large numbers assures us that $S_n \to +\infty$ with probability one, and thus for any x we have

$$P(S_n = x, \text{io}) \leqq P(S_n \leqq x, \text{io}) = 0, \tag{1}$$

where, as is customary, "io" denotes the phrase "infinitely often." From (1) we see at once that the random walk is transient, and thus for any x, y,

$$G(x, y) = \delta(x, y) + \sum_{n=1}^{\infty} P^n(x, y) < \infty, \tag{2}$$

where here and in the following, P^n denotes the nth power of the matrix P. (We shall also use the convention that P^0 is the identity matrix.) A basic result in the theory of partial sums is the so-called generalized discrete renewal theorem, which was first proved by Chung and Wolfowitz [1] as an extension of the one-sided renewal theorem of Erdos, Feller, and Pollard [3].

Theorem. *If $EX_1 > 0$, and if X_1 has an aperiodic distribution on the integers, then*

$$\lim_{y \to +\infty} G(x, y) = (EX_1)^{-1}, \tag{3}$$

$$\lim_{y \to -\infty} G(x, y) = 0, \tag{4}$$

where we interpret $(EX_1)^{-1}$ to be 0 if $EX_1 = \infty$.

By taking full advantage of the discreteness, we shall present a simple proof of this theorem using elementary probabilistic arguments. We mention in passing that a short analytic proof can be given by an appeal to a Tauberian theorem of Wiener (see [6]). One final remark seems in order before getting down to the proof. It is of course possible to have a transient random walk in which both $E[\max(0, X_1)]$, $E[-\min(0, X_1)]$ are infinite. In this case it has just recently been shown by Feller and Orey (see [5]) that $\lim_{|y| \to \infty} G(x, y) = 0$. A simplified

Received 10 December 1964.

proof can be found in [4], and, for the discrete case, in [7]. The method of proof we are about to give will not yield this result, and it seems that the more intricate methods given in [5] must be used to establish the theorem in its fullest generality.

The following well-known result will be crucial to our proof.

LEMMA 1. *The only bounded solution of the equation*

$$\sum_y P(x, y)\varphi(y) = \varphi(x) \tag{5}$$

is $\varphi(x) \equiv \varphi(0)$.

We shall not prove this result here. A simple proof (based on the diagonal procedure of Cantor) can be found in [7] (p. 276, $T1$).

For any quantity, a, let $a^+ = \max(a, 0)$ and let $a^- = \min(a, 0)$. Let $M_n = \min(S_1, \cdots, S_n)$. Then, since M_n is nonincreasing, $M = \lim_{n\to\infty} M_n$ exists, and since $EX_1 > 0$, we have from (1) that $P(M = -\infty) \leqq P(S_n \leqq 0, \text{io}) = 0$.

We shall also need the following known result (see [2]):

LEMMA 2. *If* $EX_1 > 0$, *then*

$$EM^+ = EX_1 \tag{6}$$

in the sense that both sides are finite or infinite together and always equal.

PROOF. $M_1 = X_1$, and for $n > 1$, $M_n = X_1 + \min(0, X_2, \cdots, X_2 + \cdots + X_n)$. Since the $\{X_i\}$ are independent and identically distributed, for θ real we have

$$E(e^{i\theta M_n}) = E(e^{i\theta X_1})E(e^{i\theta M_{n-1}^-}).$$

Consequently,

$$E(e^{i\theta M_n^+}) - 1 = E(e^{i\theta X_1})E(e^{i\theta M_{n-1}^-}) - E(e^{i\theta M_n^-}).$$

Since M^+ and M^- are finite with probability one, upon passing to the limit we have

$$E(e^{i\theta M^+}) - 1 = [E(e^{i\theta X_1}) - 1]E(e^{i\theta M^-}).$$

Assume first that $EX_1 < \infty$. We may then conclude from the above relation that

$$\lim_{\theta\to 0}\{[E(e^{i\theta M^+}) - 1]/i\theta\} = EX_1,$$

and since $M^+ \geqq 0$, (6) must hold. (See [7], p. 59).

An alternate derivation of (6), (again with $EX_1 < \infty$) may be made without the use of characteristic functions by employing the relation

$$E(M_n^- - M_{n-1}^-) = E(S_n/n)^-,$$

which may be derived by a simple combinatorial argument (see [2]).

Since

$$n^{-1}\sum_{k=1}^n X_k^- \leqq (S_n/n)^- \leqq 0,$$

and with probability one, $n^{-1}\sum_{k=1}^n X_k^- \to EX^-$, $(S_n/n)^- \to 0$, we have (dominated convergence) that $E(S_n/n)^- \to 0$ as $n \to \infty$. But we also have $0 \leqq M_n^+ \leqq$

$M_1^+ = X_1^+$, and so (again by dominated convergence) we have $\lim_{n\to\infty} EM_n^+ = EM^+$. Now

$$EM^+ = \lim_{n\to\infty} EM_n^+ = EX_1 + \lim_{n\to\infty} E(M_{n-1}^- - M_n^-) = EX_1 .$$

Now assume $EX_1 = +\infty$. We must then show that $EM^+ = \infty$. This is most easily accomplished by use of the following truncation argument. (This was brought to our attention by T. E. Harris.) Since $EX_1 = +\infty$, we know that given any R, however large, we may find an A such that the random variable $X_1' = X_1$ if $X_1 \leqq A$, and $X_1' = 0$ if $X_1 > A$ has expectation at least R. Clearly, $X_1' + \cdots + X_n' \leqq S_n$ and thus $(M')^+ \leqq M^+$. But then by what has just been shown above, we must have $R \leqq EX_1' = E(M')^+ \leqq EM^+$, and since R is arbitrary, we have $EM^+ = \infty$. This completes the proof of Lemma 2.

Consider now a particle which executes the random walk. For each $x \leqq 0$ let $e(x)$ be the probability that the particle, starting from x, on the first step leaves the nonpositive axis and thereafter remains on the positive axis. Define $e(x) = 0$ if $x > 0$. Then

$$\begin{aligned} e(x) &= 0, && x > 0, \\ &= P(M^+ > -x), && x \leqq 0, \end{aligned}$$

and from (6) we have

$$(7) \qquad \textstyle\sum_x e(x) = \sum_{x=0}^{\infty} e(-x) = \sum_{x=0}^{\infty} P(M^+ > x) = EM^+ = EX_1 .$$

Let $V(x)$ be the probability that our particle, starting from x, ever hits the nonpositive axis. Clearly $V(x) = 1$ if $x \leqq 0$, while for $x > 0$ we have $V(x) = P(M \leqq -x)$. The probability that a particle starting at x leaves the nonpositive axis at time $n + 1$, never to return, is $\sum_y P^n(x, y)e(y)$. Since a particle which visits the nonpositive axis at all can (with probability one) only do so a finite number of times, we must have

$$(8) \qquad V(x) = \textstyle\sum_{n=0}^{\infty} [\sum_t P^n(x, t)e(t)] = \sum_t G(x, t)e(t).$$

We are now in a position to prove the theorem. First of all we have

$$(9) \qquad 0 \leqq G(x, y) \leqq G(y, y) = G(0, 0),$$

and by definition of G we see that

$$(10) \qquad \textstyle\sum_t P(x, t)G(t, y) = G(x, y) - \delta(x, y).$$

From (9) we see that we may extract (by the Cantor diagonal procedure) a subsequence $y_n \to +\infty$ such that the limit

$$(11) \qquad \lim_{n\to\infty} G(x, y_n) = \varphi(x) \leqq G(0, 0)$$

exists for all x. If we set $y = y_n$ in (10) and then pass to the limit, we see (by dominated convergence) that φ is a bounded solution to Equation (5). Hence by Lemma 1, $\varphi(x) \equiv \alpha$ for some α. Now since $G(-y_n, t) = G(-t, y_n)$, from (8) we have

$$(12) \qquad V(-y_n) = \sum_t G(-t, y_n)e(t) = \sum_{t=0}^{\infty} G(t, y_n)e(-t).$$

Suppose $EX_1 < \infty$. Then from (7) and (12) we have (by dominated convergence)

$$1 = \lim_{n\to\infty} V(-y_n) = \lim_{n\to\infty} \sum_t G(t, y_n)e(-t) = \alpha EX_1 .$$

Hence $(EX_1)^{-1} = \alpha$. On the other hand, if $EX_1 = \infty$ then we must have $\alpha = 0$, for if $\alpha > 0$, we may choose t_0 such that $\alpha \sum_{t=0}^{t_0} e(-t) > 2$. From (12) we see

$$(13) \qquad 1 \geqq V(-y_n) \geqq \sum_{t=0}^{t_0} G(t, y_n)e(-t),$$

and thus, as $n \to \infty$, we have $1 \geqq \alpha \sum_{t=0}^{t_0} e(-t) > 2$, a contradiction. Since the only property of the sequence $\{y_n\}$ which we used in the above argument was that (11) holds, we see that if we have any other sequence $\{y_n'\}$ with this property, then we may conclude $G(x, y_n') \to (EX_1)^{-1}(= 0$ if $EX_1 = \infty)$. Hence (3) holds. To establish (4), we proceed in a similar manner. From (9) we know there is a sequence $y_n \to -\infty$ such that the limit in (11) exists for all x. From (10) we then can conclude that this limit is a constant β. For each t_0 we see from (12) that (13) holds. Since $V(-y_n) = P(M \leqq y_n)$ (for n sufficiently large) and tends to 0 as $n \to \infty$, we obtain, upon passing to the limit in (13), $0 \geqq \beta \sum_{t=0}^{t_0} e(-t)$, and since $e(-t_0) > 0$ for some t_0 (since $EM^+ > 0$), we see that β must be 0. Finally, if we had another sequence $y_n' \to -\infty$, then the same argument as used above, applied to this sequence, would show $G(x, y_n') \to 0$ for all x, and hence (4) holds. This completes the proof.

REFERENCES

[1] Chung, K. L. and Wolfowitz, J. (1952). On a limit theorem in renewal theory. *Ann. Math.* **55** 1–6.

[2] Dwass, M. (1962). A fluctuation theorem for cyclic random variables. *Ann. Math. Statist.* **33** 1450–1454.

[3] Erdos, P. W., Feller, W., and Pollard, H. (1949). A theorem on power series. *Bull. Amer. Math. Soc.* **55** 201–204.

[4] Feller, W. (1961). A simple proof of renewal theorems. *Comm. Pure Appl. Math.* **14** 285–293.

[5] Feller, W. and Orey, S. (1961). A renewal theorem. *J. Math. Mech.* **10** 619–624.

[6] Karlin, S. (1955). On the renewal equation. *Pacific J. Math.* **5** 229–257.

[7] Spitzer, F. (1964). *Principles of Random Walk.* Van Nostrand, Princeton.