STUDENT'S t IN A TWO-WAY CLASSIFICATION WITH UNEQUAL VARIANCES[1]

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1. Introduction. Since "Student" discovered the random variable t and its distribution in 1908, the statistic t has been used to test for differences between the means of two populations. The use of this statistic for testing depends on assumptions of the normality of the two underlying distributions and on the equality of their variances. If either of these assumptions does not hold, the statistic is not valid. Scheffé [2] suggested a way whereby one can validly use the statistic t for two normal populations with unequal variances, and Cochran [1] discussed how to deal with heterogeneity of error variance associated with treatments in a randomized block experiment. The methods described by Scheffé and Cochran consist of forming independent estimates of the contrast to be tested and calculating t from them. The method is generalized in this paper.

The generalization was suggested by an experiment which was designed to compare the skill of three technicians in reading pulse characteristics using two testing devices. Inspection of the data suggested that the technicians differed appreciably in their abilities to repeat readings and that one of the devices gave more reproducible readings than the other. Accordingly, the present study was undertaken.

This study investigates the two-way classification with n observations per cell, where each cell is determined by one member from each class. It assumes that an observation is a realization of a random variable which is expressed as a sum of fixed effects, the overall mean and effects for the members which determine the cell, and a random error. With respect to the fixed effects, two models are considered: one without interaction and the other with interaction. The random errors are assumed to be statistically independent and normally distributed with means zero. A variance is associated with each member of each class, and the variance of the random errors in a cell is the sum of the variances associated with the members which determine the cell. Accordingly, the random variables for all cells have the same variance within a cell, but differ from cell to cell as determined by simple additive constraints. In order to use Student's t for testing contrasts among effects, it is necessary to find error contrasts which have the same variance as the contrast to be tested and are statistically independent of it and each other. The maximum number of such contrasts is determined and a method for finding them is presented.

Received 11 February 1963; revised January 1965.

[1] Much of this study was conducted under the sponsorship of the Army Research Office (Durham), Grant No. DA-ARO(D)-31-124-G266 while the authors were with the Research Triangle Institute.

2. The model and general definitions. Consider a two-way classification with classes A and B, consisting of r rows and c columns, respectively. The general model may be written as

$$(2.1) \quad Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + U_{ijk}, \qquad (1 \leqq i \leqq r, 1 \leqq j \leqq c, 1 \leqq k \leqq n),$$

where the U_{ijk} are independently distributed as $N(0, \sigma^2_{\alpha_i} + \sigma^2_{\beta_j})$, and μ, α_i, β_j, $(\alpha\beta)_{ij}$ are fixed effects associated respectively with the general mean, the ith row, the jth column, and the row-column interaction, with $\sum_i \alpha_i = 0$, $\sum_j \beta_j = 0$, $\sum_i (\alpha\beta)_{ij} = \sum_j (\alpha\beta)_{ij} = 0$. In Section 3, the general model is specialized by taking $(\alpha\beta)_{ij} = 0$ for all i, j, but in Section 4 the interaction effects are retained.

The following definitions will be used in the sequel.

DEFINITION 1. A general linear function is defined as $L = \sum_{i,j} L_{ij}$, where $L_{ij} = \sum_k c_{ijk} Y_{ijk}$.

DEFINITION 2. Replacing a suffix in Y_{ijk} by a dot will denote the result of averaging over all values of the suffix; thus cell, row, and column averages will be denoted by $Y_{ij\cdot}$, $Y_{i\cdot\cdot}$, and $Y_{\cdot j\cdot}$, respectively.

DEFINITION 3. The notation

$$C_s(A) = \sum a_{si} Y_{i\cdot\cdot}, \quad \sum_i a_{si} = 0,$$
$$C_t(B) = \sum_j b_{tj} Y_{\cdot j\cdot}, \quad \sum_j b_{tj} = 0,$$

will be used for contrasts between row and column means, respectively.

DEFINITION 4. The notation

$$C_{st}(AB) = \sum_{i,j} a_{si} b_{tj} Y_{ij\cdot}, \qquad \sum_i a_{si} = 0, \qquad \sum_j b_{tj} = 0,$$

will denote an interaction contrast.

DEFINITION 5. The notation

$$C_v(E) = \sum_{i,j,k} e_{vijk} Y_{ijk}, \quad \sum_{i,j,k} e_{vijk} = 0, \; E[C_v(E)] = 0,$$

will denote a contrast which belongs to error.

3. Linear functions belonging to error for the first model. The following theorems develop the form of an estimator of variance to be used in t for testing the null hypotheses that $E[C_s(A)] = 0$ and that $E[C_t(B)] = 0$. They also discuss the number of degrees of freedom possible for testing such hypotheses.

Any linear function $C(E)$ which can be used to test the hypothesis $E[C_s(A)] = 0$ must satisfy conditions (i) and (ii) and any two such functions $C_v(E)$ and $C_{v'}(E)$ must satisfy condition (iii):

$$(3.1) \quad \begin{aligned} &\text{(i) } \operatorname{Cov}[C(A), C(E)] = 0, \\ &\text{(ii) } V[C(A)] = KV[C(E)], \quad K > 0, \text{ independently of } \sigma^2_{\alpha_i} \text{ and } \sigma^2_{\beta_j}, \\ &\text{(iii) } \operatorname{Cov}[C_v(E), C_{v'}(E)] = 0, \qquad v \neq v'. \end{aligned}$$

For the hypothesis $E[C_t(B)] = 0$, $C(A)$ in (i) and (ii) is of course replaced by $C(B)$.

A basis for all contrasts is defined by orthogonal sets of contrasts between row and column means, $\{C_s(A)\}$, $(1 \leqq s \leqq r - 1)$ and $\{C_t(B)\}$, $(1 \leqq t \leqq c - 1)$, the corresponding orthogonal set of row-column interaction contrasts, $\{c_{st}(AB)\}$, and sets of within-cell contrasts, $\{C_{ij}^{(w)}\}$, $(1 \leqq i \leqq r, 1 \leqq j \leqq c)$. Clearly, the coefficients of $C_s(A)$ and $C_t(B)$ in an error contrast $C(E)$ are zero. It therefore follows that

$$C(E) = L\{C_{st}(AB)\} + L\{C_{ij}^{(w)}\}, \tag{3.2}$$

which is summarized in Theorem 1.

THEOREM 1. *Every linear function which belongs to error is a linear combination of interaction contrasts and within-cell contrasts, as is stated in* (3.2).

Consideration will now be given to the set of interaction contrasts which are uncorrelated with $C_s(A)$, and a basis for this set will be exhibited.

THEOREM 2. *The number of degrees of freedom carried by the interaction contrasts which are uncorrelated with* $C_s(A)$ *is* $(r - 2)(c - 1)$. *A basis for this set of contrasts is* $C_{s't}(AB)$ *where* $s' = 1, \cdots, s - 1, s + 1, \cdots, r - 1$.

The maximum number of such contrasts which are mutually uncorrelated will in general be less than $(r - 2)(c - 1)$, *but will always be at least* $\min(r - 2, c - 1)$.

PROOF. Consider the condition (3.1), (i) as applied to interaction contrasts. For $C_s(A)$ and $C_{st}(AB)$,

$$\begin{aligned}\text{Cov}\,[C_s(A), C_{st}(AB)] &= \text{Cov}\,(\textstyle\sum_i a_{si}Y_{i\cdot\cdot}, \sum_{i,j} a_{si}b_{tj}Y_{ij\cdot}) \\ &= \textstyle\sum_i a_{si}^2[(\sum_j b_{tj}\sigma_{\alpha_i}^2 + \sum_j b_{tj}\sigma_{\beta_j}^2)/nc].\end{aligned}$$

But, since $\sum_j b_{tj} = 0$,

$$\text{Cov}\,[C_s(A), C_{st}(AB)] = \textstyle\sum_i a_{si}^2 \sum_j b_{tj}\sigma_{\beta_j}^2/nc,$$

which is not equal to zero as required by (3.1), (i).

Now, consider

$$\begin{aligned}\text{Cov}\,[C_s(A), C_{s't}(AB)] &= \textstyle\sum_i a_{si}a_{s'i}\sum_j b_{tj}(\sigma_{\alpha_i}^2 + \sigma_{\beta_j}^2)/nc \\ &= \textstyle\sum_i a_{si}a_{s'i}\sum_j b_{tj}\sigma_{\beta_j}^2/nc.\end{aligned}$$

Now, since $\sum_i a_{si}a_{s'i} = 0$, [Cov $C_s(A), C_{s't}(AB)] = 0$, as required by (3.1), (i). Thus, the number of degrees of freedom carried by the interaction contrasts uncorrelated with $C_s(A)$ is $(r - 2)(c - 1)$. Because the $C_{s't}(AB)$ contrasts are defined to be mutually orthogonal, they form a basis for the set of interaction contrasts uncorrelated with $C_s(A)$.

Finally, consider the covariance of any two interaction contrasts, $C_{s't}(AB)$ and $C_{s''t'}(AB)$:

$$\begin{aligned}\text{Cov}\,[C_{s't}(AB), C_{s''t'}(AB)] &= \textstyle\sum_{i,j} a_{s'i}b_{tj}a_{s''i}b_{t'j}(\sigma_{\alpha_i}^2 + \sigma_{\beta_j}^2)/n \\ &= (\textstyle\sum_i a_{s'i}a_{s''i}\sigma_{\alpha_i}^2 \sum_j b_{tj}b_{t'j} \\ &\quad + \textstyle\sum_j b_{tj}b_{t'j}\sigma_{\beta_j}^2 \sum_i a_{s'i}a_{s''i})/n,\end{aligned}$$

which is not necessarily zero unless $s' \neq s''$ and $t' \neq t$. But this implies that the number of mutually uncorrelated interaction contrasts which are uncorrelated with $C_s(A)$ does not exceed min $(r - 2, c - 1)$.

THEOREM 3. *The number of degrees of freedom carried by the interaction contrasts which are uncorrelated with* $C_t(B)$ *is* $(r - 1)(c - 2)$. *A basis for this set of contrasts is* $C_{st'}(AB)$, $t' = 1, 2, \cdots, t - 1, t + 1, \cdots, c - 1$. *However, the maximum number of such contrasts which are mutually uncorrelated will in general be less than* $(r - 1)(c - 2)$, *but will always be at least* min $(r - 1, c - 2)$.

PROOF. The proof of this theorem is similar to that of Theorem 2 with $C_s(A)$ replaced by $C_t(B)$ and $C_{s't}(AB)$ replaced by $C_{st'}(AB)$.

THEOREM 4. *It is necessary and sufficient that an interaction contrast* $C_{s't}(AB)$ *used to test the hypothesis that* $E[C_s(A)] = 0$ *be such that* $a_{s'i} = \pm Ra_{si}$ *for all i, where R is a constant, and that* $b_{tj} = \pm b$ *for all j.*

PROOF. If $C_{s't}(AB)$ can be used to test the hypothesis that $E[C_s(A)] = 0$, it is necessary from (3.1), (ii) that

$$V[C_s(A)] = KV[C_{s't}(AB)], \tag{3.3}$$

where K is a positive constant.

Now,

$$\begin{aligned} V[C_{s't}(AB)] &= V[\textstyle\sum_{i,j} a_{s'i}b_{tj}Y_{ij\cdot}] \\ &= \textstyle\sum_{i,j} a_{s'i}^2 b_{tj}^2(\sigma_{\alpha_i}^2 + \sigma_{\beta_j}^2)/n, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} V[C_s(A)] &= V[\textstyle\sum_i a_{si}Y_{i\cdot\cdot}] \\ &= \textstyle\sum_i a_{si}^2(\sigma_{\alpha_i}^2 + \sum_j \sigma_{\beta_j}^2/c)/nc. \end{aligned} \tag{3.5}$$

From (3.3), (3.4), and (3.5), the coefficients of $\sigma_{\alpha_i}^2$ and $\sigma_{\beta_j}^2$ in the two variances must be constant multiples of each other. For $\sigma_{\alpha_i}^2$ this yields

$$a_{si}^2/c = Ka_{s'i}^2\textstyle\sum_j b_{tj}^2, \tag{3.6}$$

and for $\sigma_{\beta_j}^2$ it yields

$$\textstyle\sum_i a_{si}^2/c^2 = K(\sum_i a_{s'i}^2)b_{tj}^2. \tag{3.7}$$

Summing over j in (3.7) gives

$$\textstyle\sum_i a_{si}^2/c = K\sum_i a_{s'i}^2\sum_j b_{tj}^2. \tag{3.8}$$

Dividing (3.6) by (3.8) shows that $a_{si}^2/\sum_i a_{si}^2 = a_{s'i}^2/\sum_i a_{s'i}^2$, which implies that

$$a_{s'i} = \pm Ra_{si} \quad \text{for all } i. \tag{3.9}$$

Substituting (3.9) in (3.7) gives

$$c^{-2} = KR^2b_{tj}^2. \tag{3.10}$$

which implies that $b_{tj} = \pm b$ for all j. The conditions, $a_{s'i} = \pm R a_{si}$ and $b_{tj} = \pm b$, are sufficient since they satisfy (3.3).

THEOREM 5. *It is necessary and sufficient that an interaction contrast* $C_{st}(AB)$ *used to test the hypothesis that* $E[C_t(B)] = 0$ *be such that* $b_{t'j} = \pm M b_{tj}$ *for all* j, *where* M *is a constant, and that* $a_{si} = \pm a$ *for all* i.

PROOF. The proof of this theorem is similar to that of Theorem 4, replacing $C_{s't}(AB)$ by $C_{st'}(AB)$ and $C_s(A)$ by $C_t(B)$.

When the number of columns is odd, it is impossible to satisfy the condition that $b_{tj} = \pm b$ for all j. Accordingly, there is no suitable contrast uncorrelated with $C_s(A)$. A similar remark applies when the number of rows is odd.

The following theorems are stated without proof:

THEOREM 6. *The number of degrees of freedom carried by the mutually uncorrelated interaction contrasts which can be used to test the null hypothesis that* $E[C_t(A)] = 0$, *when* $p \leqq r - 2$ *of the* $a_{si} = 0$, *does not exceed* $\min(r - 2 - p, c - 1)$.

THEOREM 7. *The number of degrees of freedom carried by the mutually uncorrelated interaction contrasts which can be used to test the null hypothesis that* $E[C_t(B)] = 0$, *when* $q \leqq c - 2$ *of the* $b_{tj} = 0$, *does not exceed* $\min(r - 1, c - 2 - q)$.

4. Linear functions belonging to error for both models. Only one major difference occurs when the interaction term is added to the model. This difference concerns the possible linear functions which belong to error. With the interaction term included, the parameter space is completely filled, and only linear functions from the error space may be used to test the null hypotheses that $E[C_s(A)] = 0$, $E[C_t(B)] = 0$, and $E[C_{st}(AB)] = 0$, i.e., only linear functions of the random variables within the cells may be used. Thus, the theorems concerning interaction contrasts do not hold for this model.

The rest of this paper deals with the properties of the error contrasts which are linear functions of the random variables within the cells. These properties are valid for both models discussed.

THEOREM 8. *There are at most* $[(n - 1)\min(r, c)]$ *statistically independent linear functions of contrasts among the random variables within the cells which belong to error and can be used to test the hypotheses that* $E[C_s(A)] = 0$, $E[C_t(B)] = 0$, *and* $E[C_{st}(AB)] = 0$.

PROOF. Consider testing the null hypothesis that $E[C_s(A)] = 0$. Any linear function $C(E)$ which can be used to test this hypothesis must satisfy the conditions stated in (3.1). From the definition of a contrast among the n random variables within a cell, it follows that $C(E)$ is uncorrelated with their sum, and hence with $C_s(A)$. Accordingly, condition (3.1), (i) is met.

We shall consider linear functions of the form $C^{(w)}(E) = \sum_{i,j} k_{ij}^{(w)} C_{ij}^{(w)}(E)$, where the $k_{ij}^{(w)}$ are constant coefficients and the coefficient $e_k^{(w)}$ of Y_{ijk} in $C_{ij}^{(w)}(E)$ does not depend on i and j. The variance of $C^{(w)}(E)$ is

$$\begin{aligned} V[C^{(w)}(E)] &= \textstyle\sum_{i,j} (k_{ij}^{(w)})^2 V[C_{ij}^{(w)}(E)] \\ &= \textstyle\sum_k (e_k^{(w)})^2 \sum_{i,j} (k_{ij}^{(w)})^2 (\sigma_{\alpha_i}^2 + \sigma_{\beta_j}^2). \end{aligned}$$

Because the variance of $C_s(A)$ is

$$V[C_s(A)] = \textstyle\sum_i a_{si}^2(\sigma_{\alpha_i}^2 + \sum_j \sigma_{\beta_j}^2/c)/nc, \tag{4.1}$$

it is necessary by condition (3.1), (ii) that

$$\textstyle\sum_j (k_{ij}^{(w)})^2 = Ka_{si}^2/nc, \qquad i = 1, \cdots, r, \tag{4.2}$$

and

$$\textstyle\sum_i (k_{ij}^{(w)})^2 = K\sum_i a_{si}^2/nc^2, \qquad j = 1, \cdots, c.$$

The covariance of two such functions is

$$\text{Cov }[C^{(1)}(E),\ C^{(2)}(E)] = \textstyle\sum_k (e_k^{(w)})^2 \sum_{i,j} k_{ij}^{(1)} k_{ij}^{(2)} (\sigma_{\alpha_i}^2 + \sigma_{\beta_j}^2), \tag{4.3}$$

using the same set of coefficients $\{e_k^{(w)}\}$ for the within-cell comparisons. But by condition (3.1), (iii),

$$\textstyle\sum_j k_{ij}^{(1)} k_{ij}^{(2)} = \sum_i k_{ij}^{(1)} k_{ij}^{(2)} = 0, \quad \text{for all } i \text{ and } j. \tag{4.4}$$

It, therefore, follows that there are at most min (r, c) such functions based on a given set of coefficients $\{e_k^{(w)}\}$. But, because of (4.2) and the possibility that $C_s(A)$ may be a contrast for which some of the a_{si} are zero, there may be fewer than min (r, c) such functions. Because there are $(n - 1)$ mutually orthogonal sets of coefficients $\{e_k^{(w)}\}$ for the within-cell comparison, the theorem follows.

However, for $r = 2^p$ and $c = 2^q$, $0 < p \leq q$, p and q positive integers, and all $a_{si} \neq 0$, more specific results can be stated. Let k_i^T denote the row vector $[k_{11}^{(i)}, \cdots, k_{1c}^{(i)}, \cdots, k_{r1}^{(i)}, \cdots, k_{rc}^{(i)}]$. Also, write $E[Y] = AX$ for the equations of expectation associated with a factorial design in which all factors have two levels. In particular, A is an orthogonal matrix with elements 1 and -1. Let the coefficients of the $Y_{i\cdot\cdot}$ in $C_s(A)$ be the elements in any column of the matrix A for the design of order 2^p, except the column which corresponds to the mean. Then the a_{si}^2 are 1 and (4.1) reduces to $[\sum_i \sigma_{\alpha_i}^2 + (r/c)\sum_j \sigma_{\beta_j}^2]/nc$. Also, (4.2) becomes $\sum_j (k_{ij}^{(w)})^2 = K/nc$ and $\sum_i (k_{ij}^{(w)})^2 = Kr/nc^2$. Clearly, apart from the constant K, these conditions are met for $k_{ij}^{(w)} = \pm(nc^2)^{-\frac{1}{2}}$.

It will now be shown how to construct r functions with coefficients $\pm(nc^2)^{-\frac{1}{2}}$ which satisfy (4.4). From the design of order 2^p, select the r column vectors, multiply them by $(nc^2)^{-\frac{1}{2}}$ and denote them by $\alpha_1, \cdots, \alpha_r$. Similarly, from the design of order 2^q, construct column vectors $\beta_1, \cdots, \beta_r$. Then form the direct products $\alpha_i \times \beta_i$, such that each element of α_i is multiplied by β_i. Put $k_i^T = \alpha_i \times \beta_i$, $i = 1, \cdots, r$. Now clearly the inner product of k_i and k_j satisfies (4.4). There are $(n - 1)$ mutually orthogonal sets $\{e_k^{(w)}\}$, and hence there are $[(n - 1)$ min $(r, c)]$ mutually uncorrelated linear functions.

A similar construction can be made from Hadamard matrices.

The above argument can also be used for the hypotheses that $E[C_t(B)] = 0$ and that $E[C_{st}(AB)] = 0$. The $k_{ij}^{(w)}$ become $\pm(nr^2)^{-\frac{1}{2}}$ and $n^{-\frac{1}{2}}$, respectively.

Further, for $r = c$ (even), $(n - 1)r$ mutually uncorrelated linear functions are available to test whether $E[C_s(A)] = 0$ when the coefficients in $C_s(A)$ are 1 or -1. This is readily seen by considering matrices $K^{(w)}$ of the $k_{ij}^{(w)}$. Let the matrix $K^{(1)}$ contain one in every position and the matrix $K^{(u)}$, $(u = 2, \cdots, r)$ be a cyclic matrix with elements in the first row as follows: the first $(u - 2)$ elements are zero, the $(u - 1)$st element is $(r - u + 1)$, and the remaining elements are -1. Clearly, the elements in these matrices satisfy (4.4) and by multiplication of the $K^{(w)}$ by suitable scalars, they can be made to satisfy (4.2). In addition, if r and c are even, and r is a factor of c, there are $(n - 1)r$ mutually uncorrelated linear functions which can be constructed by forming matrices $K^{(1)}$ and $K^{(u)}$ by repetition of the above matrices c/r times.

There remain unsettled problems which will not be discussed in this paper. Among these are what is the number of uncorrelated error contrasts when r and c are of the above forms but the coefficients in $C_s(A)$ are not all ± 1, and when r and c are not of the above forms? These problems do not appear to be as readily solvable as those discussed above, and might serve as a subject for a future investigation.

5. Example. The example which will be discussed in this section makes use of part of the data which were collected in a study sponsored in 1951–52 by the American Society for Testing Materials Task Group 2, Sub-Committee B-5, on Fiber Content of Part Wool Blankets. Among the objectives of the study were the comparison of the sulfuric acid and the sodium hydroxide methods of determining the present wool in a blanket and the comparison of participating laboratories. Mr. J. M. Cameron and the second author, both of whom at that time were members of the Statistical Engineering Laboratory of the National Bureau of Standards, prepared a Latin Square design, which provided for the participation of four laboratories, each of which used the two methods of test

TABLE 1

Method	Laboratory 1		Laboratory 2	
S	2.1	2.7	0.3	1.3
	0.8	6.3	−0.3	0.3
	1.0	0.6	0.2	0.0
	1.3	0.3	0.0	−0.2
	9.8	0.2	0.3	−0.5
	$\bar{x} = 2.51$		$\bar{x} = 0.14$	
	$s^2 = 9.81$		$s^2 = 0.24$	
A	0.7	4.8	0.2	1.0
	0.6	7.6	0.5	0.1
	0.5	4.9	0.2	0.2
	0.8	1.4	−0.4	−0.4
	3.3	4.6	0.4	0.2
	$\bar{x} = 2.92$		$\bar{x} = 0.20$	
	$s^2 = 6.14$		$s^2 = 0.17$	

In the notation of this paper, $r = c = 2$ and $n = 10$.

at two different times. Altogether there were 256 determinations. For the present purposes only two laboratories are used, and only one time and ten determinations for each laboratory and method.

The per cent wool was nominally 25 per cent. The readings, after being coded by subtracting 24, were as presented in Table 1. The sulfuric acid method is denoted by S and the sodium hydroxide method by A. The sample mean $\bar{x}$ and estimated variance s^2 is given for the ten determinations in each cell.

Because there may be interaction between the laboratories and the methods, the model of Section 4 with the interaction effect retained appears appropriate. For the case of two rows and two columns and general n, there are $2(n - 1)$ mutually uncorrelated linear functions, which can be chosen so that their squares have expectation $(\sigma^2_{\alpha_1} + \sigma^2_{\alpha_2} + \sigma^2_{\beta_1} + \sigma^2_{\beta_2})$. The sum of their squares is given by the convenient computing formula

$$S^2 = \textstyle\sum_{i,j} S_{ij,ij} + 2(S_{11,22} + S_{12,21}),$$

where

$$S_{i_1j_1,i_2j_2} = \textstyle\sum_k (Y_{i_1j_1k} - Y_{i_1j_1\cdot})(Y_{i_2j_2k} - Y_{i_2j_2\cdot}).$$

Accordingly, $S^2/(\sigma^2_{\alpha_1} + \sigma^2_{\alpha_2} + \sigma^2_{\beta_1} + \sigma^2_{\beta_2})$ has the χ^2-distribution with $2(n - 1)$ degrees of freedom. Here, $S_{11,11} = 88.25$, $S_{12,12} = 2.18$, $S_{21,21} = 55.30$, $S_{22,22} = 1.50$, $S_{11,22} = 5.54$, $S_{12,21} = 5.94$, and $S^2 = 170.19$. The row and column means are $Y_{1\cdot\cdot} = 1.32$, $Y_{2\cdot\cdot} = 1.56$, $Y_{\cdot 1\cdot} = 2.72$, and $Y_{\cdot 2\cdot} = 0.17$. The t-statistic for testing the null hypothesis that $\alpha_1 = \alpha_2$ is $t_{2(n-1)} = 2n^{\frac{1}{2}}(n - 1)^{\frac{1}{2}}(Y_{1\cdot\cdot} - Y_{2\cdot\cdot})/S$ with similar formulae for the hypotheses $\beta_1 = \beta_2$ and $(\alpha\beta) = 0$. The numerical values of t are 0.35, 3.61, and 0.25, respectively, indicating no difference between methods when averaged over laboratories, but a significant difference between laboratories when averaged over methods, and a non-significant interaction.

6. Contributions of the authors. The original version of this paper was the thesis of the first author in fulfilling the requirements for the Master's degree at North Carolina State University. The present version is a considerable revision of the first version and contains additional material due to the second author, particularly in Sections 4 and 5.

7. Acknowledgment. The authors wish to thank the referee for his very careful and competent reading of several versions of the paper, which led to the elimination of errors and substantial reorganization and extension.

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