# GENERALIZED RIGHT ANGULAR DESIGNS[1]

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**1. Summary and introduction.** Investigations of association schemes for three or more associate class PBIB designs have been limited mainly to the works of Vartak [17], Raghavarao [5], Roy [9], Singh and Shukla [15], Raghavarao and Chandrasekhararao [8] and Tharthare [16]. In this article we generalize the right angular association scheme introduced by the author in [16] to study the combinatorial properties, construction and nonexistence of further class of four associate class designs. In Section 4, a simplified method of analysis of these designs is given, while Section 5 deals with the method of constructing $v \times s^m$ balanced asymmetrical factorial design in $vs^{m-1}$ plots, where $s$ is a prime or a prime power. It can be seen that this balanced asymmetrical design is nothing but a particular case of the four associate class design introduced in this paper and hence its analysis can be completed by the method of Section 4. Some methods of construction discussed in Section 6 give a new way of arranging an $s^3$ factorial experiment in blocks of sizes different from $s$ and $s^2$, with better efficiency than the usual three dimensional lattice designs [4]. For convenience we denote PBIB designs with the generalized right angular association scheme by "GRAD".

**2. Definition of association scheme and parameters of generalized right angular designs.** In the generalized right angular designs we have $v = pls$ treatments which are arranged into a rectangular array as follows. The array shall consist of $p$ rows and $l$ columns of subgroups, each subgroup consisting of precisely $s$ treatments, each treatment appearing in one and only one subgroup. A column of $p$ subgroups will be called a group, a group consists of $sp$ treatments. Subgroups lying in the same row will be referred to as occupying the same positions within the groups.

Group 1

| | | | |
|---|---|---|---|
| $1$ | $2$ | $\cdots$ | $s$ |
| $s+1$ | $s+2$ | $\cdots$ | $2s$ |
| $\vdots$ | $\vdots$ | | $\vdots$ |
| $s(p-1)+1$ | $s(p-1)+2$ | $\cdots$ | $sp$ |

Group 2

| | | | |
|---|---|---|---|
| $sp+1$ | $sp+2$ | $\cdots$ | $s(p+1)$ |
| $s(p+1)+1$ | $s(p+1)+2$ | $\cdots$ | $s(p+2)$ |
| $\vdots$ | $\vdots$ | | $\vdots$ |
| $s(2p-1)+1$ | $s(2p-1)+2$ | $\cdots$ | $2sp$ |

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Group $l$

$$\begin{array}{cccc} sp(l-1)+1 & sp(l-1)+2 & \cdots & sp(l-1)+s \\ sp(l-1)+s+1 & sp(l-1)+s+2 & \cdots & sp(l-1)+2s \\ \vdots & \vdots & & \vdots \\ s(pl-1)+1 & s(pl-1)+2 & \cdots & lsp \end{array}$$

We define the four associates of a particular treatment $\phi$ as follows:

(1) Treatments other than $\phi$ occurring in the same subgroup with $\phi$ are its first associates.

(2) Treatments occurring in different subgroups than $\phi$ but in the same group with $\phi$ are the second associates of $\phi$.

(3) Treatments occurring in subgroups occupying the same positions within the group as the subgroup which contains $\phi$ are the third associates of $\phi$.

(4) The remaining treatments are the fourth associates of $\phi$.

We designate the subgroup occurring in the $\alpha$th row and the $\beta$th column of the rectangular array of the association scheme of GRAD by $(\alpha, \beta)$ and $\gamma$th treatment in it by $(\alpha, \beta, \gamma)$ $\{\alpha = 1, 2, \cdots, p; \beta = 1, 2, \cdots, l; \gamma = 1, 2, \cdots, s\}$.

Let $n_i$ be the number of $i$th associates of a given treatment and let $\lambda_i$ be the number of blocks in which a pair of $i$th associates appear $(i = 1, 2, 3, 4)$.

Clearly $v = pls$; $\sum_{i=1}^{4} n_i = v - 1$; $\sum_{i=1}^{4} n_i\lambda_i = r(k-1)$, where $n_1 = s - 1$, $n_2 = s(p-1)$, $n_3 = s(l-1)$, $n_4 = s(l-1)(p-1)$ and

$$(p_{ij}^1) = \begin{bmatrix} s-2 & 0 & 0 & 0 \\ & s(p-1) & 0 & 0 \\ & & s(l-1) & 0 \\ & & & s(l-1)(p-1) \end{bmatrix}$$

$$(p_{ij}^2) = \begin{bmatrix} 0 & s-1 & 0 & 0 \\ & s(p-2) & 0 & 0 \\ & & 0 & s(l-1) \\ & & & s(l-1)(p-2) \end{bmatrix}$$

$$(p_{ij}^3) = \begin{bmatrix} 0 & 0 & s-1 & 0 \\ & 0 & 0 & s(p-1) \\ & & s(l-2) & 0 \\ & & & s(p-1)(l-2) \end{bmatrix}$$

$$(p_{ij}^4) = \begin{bmatrix} 0 & 0 & 0 & s-1 \\ & 0 & s & s(p-2) \\ & & 0 & s(l-2) \\ & & & s(l-2)(p-2) \end{bmatrix}.$$

Now it can be seen that for nondegenerate designs, lower positive integral bounds on $l$, $p$ and $s$ are given by $l, p, s \geqq 2$ and degenerate cases can be given as follows:

(a) If $s = 1$, then $n_1 = 0$ and the generalized right angular association scheme degenerates into rectangular association scheme [17].

(b) 1. If $p = 1$, then $n_2 = n_4 = 0$ and the generalized right angular association scheme degenerates into GD association scheme.

2. If $l = 1$, then $n_3 = n_4 = 0$ and the generalized right angular association scheme degenerates into GD association scheme.

**3. Characterization of generalized right angular designs.** Let $n_{ij} = 1$, if the $i$th treatment occurs in the $j$th block; $n_{ij} = 0$, otherwise. Then the $v \times b$ matrix $N = (n_{ij})$ is known as the incidence matrix of the GRAD. From the definition of GRAD, we can see that

$$\sum_{j=1}^{b} n_{ij}^2 = r, \qquad i = 1, 2, \cdots, v.$$

$$\sum_{j=1}^{b} n_{ij}n_{i'j} = \lambda_1, \lambda_2, \lambda_3 \text{ or } \lambda_4,$$

according as $i$ and $i'$ are 1st, 2nd, 3rd or 4th associates, $i \neq i' = 1, 2, \cdots, v$.

Now, by suitably numbering the treatments, we have

$$NN' = I_l \times (C - D) + E_{ll} \times D, \tag{3.1}$$

where $C$ and D are $ps \times ps$ square matrices given by

$$\begin{aligned} C &= I_p \times (A - B) + E_{pp} \times B, \\ D &= I_p \times (F - H) + E_{pp} \times H, \\ A &= I_s \times (r - \lambda_1) + \lambda_1 E_{ss}, \qquad B = \lambda_2 E_{ss}, \\ F &= \lambda_3 E_{ss}, \qquad H = \lambda_4 E_{ss}, \end{aligned} \tag{3.2}$$

where $I_s$ is an identity matrix of order $s$; $E_{ss}$ is a square matrix of order $s$ with positive unit elements everywhere and $\times$ denotes the Kronecker product of matrices. The order of $NN'$ is $pls$. The determinant of $NN'$ can be obtained as $\theta_0^{\alpha_0} \cdot \theta_1^{\alpha_1} \cdot \theta_2^{\alpha_2} \cdot \theta_3^{\alpha_3} \cdot \theta_4^{\alpha_4}$ where

$$\begin{aligned} \theta_0 &= rk, \\ \theta_1 &= r - \lambda_1 + s(\lambda_1 - \lambda_2) + s(l - 1)(\lambda_3 - \lambda_4), \\ \theta_2 &= r - \lambda_1, \\ \theta_3 &= r - \lambda_1 + s\{\lambda_1 - \lambda_3 + (p - 1)(\lambda_2 - \lambda_4)\}, \\ \theta_4 &= r - \lambda_1 + s(\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4). \end{aligned} \tag{3.3}$$

It can be observed that $\theta_0$, $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ are distinct characteristic roots of $NN'$ with respective multiplicities $\alpha_0 = 1$, $\alpha_1 = (p - 1)$, $\alpha_2 = pl(s - 1)$, $\alpha_3 = (l - 1)$ and $\alpha_4 = (p - 1)(l - 1)$. We know from the result of Connor and Clatworthy [3] that the characteristic roots of $NN'$ cannot be negative for an existing design. Thus, we have the following theorem.

THEOREM 3.1. *A necessary condition for the existence of a generalized right angular design is that* $\theta_i \geqq 0$ $(i = 1, 2, 3, 4)$.

The designs with the following parameters violate the above necessary condition and hence are impossible. The reason of impossibility is shown in parenthesis against the parameters.

| $v$ | $l$ | $p$ | $s$ | $b$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 2 | 3 | 2 | 8 | 4 | 6 | 4 | 0 | 2 | 3 | $(\theta_3 < 0)$ |
| 16 | 2 | 4 | 2 | 10 | 5 | 8 | 3 | 5 | 1 | 0 | $(\theta_4 < 0)$ |
| 24 | 4 | 3 | 2 | 20 | 10 | 12 | 8 | 6 | 3 | 5 | $(\theta_1 < 0)$ |
| 24 | 4 | 3 | 2 | 32 | 12 | 9 | 8 | 1 | 8 | 3 | $(\theta_3 < 0)$ |
| 27 | 3 | 3 | 3 | 18 | 8 | 12 | 5 | 3 | 2 | 4 | $(\theta_1 < 0)$ |
| 36 | 3 | 4 | 3 | 24 | 10 | 15 | 13 | 2 | 1 | 5 | $(\theta_2 < 0)$ |

**4. Analysis.** With the usual intrablock model, the normal equations giving the column vectors of the intrablock estimates of the treatment effects $\hat{\mathbf{t}}$ are

$$\mathbf{Q} = C\hat{\mathbf{t}}, \tag{4.1}$$

where $\mathbf{Q} = \mathbf{T} - k^{-1}N\mathbf{B}$; $C = rI_v - k^{-1}NN'$, $\mathbf{T}$ and $\mathbf{B}$ being the column vectors of the treatment totals and block totals respectively.

Following Shah [10], the solution of the reduced normal equation can be written as

$$\hat{t} = (C + aE_{vv})^{-1}\mathbf{Q} = \textstyle\sum_{i=0}^{4} (A_i/\phi_i)\mathbf{Q}, \tag{4.2}$$

where $\phi_0 = av$, $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$ are the characteristic roots of $C + aE_{vv}$ where $\phi_i = (r - \theta_i/k)$ $i = 1, 2, 3, 4$ and normalised orthogonal vectors corresponding to $\phi_0$, $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$ are

$$v^{-\frac{1}{2}}E_{v,1};\; l^{-\frac{1}{2}}E_{l,1} \times \begin{bmatrix} 2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [p(p-1)]^{-\frac{1}{2}} \\ -2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [p(p-1)]^{-\frac{1}{2}} \\ 0 & -2(2.3)^{-\frac{1}{2}} & \cdots & \\ \cdot & 0 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & -(p-1)[p(p-1)]^{-\frac{1}{2}} \end{bmatrix} \times s^{-\frac{1}{2}}E_{s,1}; \tag{4.3}$$

$$I_{pl} \times \begin{bmatrix} 2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [s(s-1)[^{-\frac{1}{2}} \\ -2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [s(s-1)]^{-\frac{1}{2}} \\ 0 & -2(2.3)^{-\frac{1}{2}} & \cdots & \\ \cdot & 0 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & -(s-1)[s(s-1)]^{-\frac{1}{2}} \end{bmatrix};$$

$$\begin{bmatrix} 2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [l(l-1)]^{-\frac{1}{2}} \\ -2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [l(l-1)]^{-\frac{1}{2}} \\ 0 & -2(2.3)^{-\frac{1}{2}} & \cdots & \\ \cdot & 0 & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & -(l-1)[l(l-1)]^{-\frac{1}{2}} \end{bmatrix} \times (sp)^{-\frac{1}{2}}E_{sp,1};$$

$$\begin{bmatrix} 2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [l(l-1)]^{-\frac{1}{2}} \\ -2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [l(l-1)]^{-\frac{1}{2}} \\ 0 & -2(2.3)^{-\frac{1}{2}} & \cdots & \\ \cdot & & & \cdot \\ \cdot & 0 & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & -(l-1)\,[l(l-1)]^{-\frac{1}{2}} \end{bmatrix}$$

$$\times \begin{bmatrix} 2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [p(p-1)]^{-\frac{1}{2}} \\ -2^{-\frac{1}{2}} & (2.3)^{-\frac{1}{2}} & \cdots & [p(p-1)]^{-\frac{1}{2}} \\ 0 & -2(2.3)^{-\frac{1}{2}} & \cdots & \\ \cdot & & & \cdot \\ \cdot & 0 & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & -(p-1)\,[p(p-1)]^{-\frac{1}{2}} \end{bmatrix} \times s^{-\frac{1}{2}} E_{s,1} .$$

Now the mutually orthogonal symmetric idempotent matrices $A_0$, $A_1$, $A_2$, $A_3$ and $A_4$ corresponding to roots $\phi_0$, $\phi_1$, $\phi_2$, $\phi_3$ and $\phi_4$ can be obtained and so (4.2) can be written as

$$(4.4)\quad \hat{t}_i = \{[(s-1)Q_i - H_i]/s\phi_2 + (\phi_1 - \phi_4)(Z_i + J_i)/sl\phi_1\phi_4 + [J_i(\phi_4 - \phi_3) + (Q_i + H_i)[(p-1)\phi_3 + \phi_4]]/sp\phi_3\phi_4\},$$

where $Q_i$ denotes the $i$th adjusted treatment totals and $H_i$, $J_i$ and $Z_i$ denote the sum of the adjusted 1st, 2nd and 4th associate treatment totals of the $i$th treatment. The treatment sum of squares adjusted for blocks can be calculated in the usual manner, from the solution of the normal equations (4.4) as given in Kempthorne [4]. It can be seen that the variance of the elementary treatment contrasts and average variance of all elementary treatment contrasts is given in terms of characteristic roots $\phi_i$'s of $C + aE_{vv}$ as

$$V(t_i - t_i') = 2\sigma^2/\phi_2 \quad \text{when the } i\text{th and } i'\text{th treatments are 1st associates,}$$

(4.5)

$$= 2\{p/v\phi_1 + (s-1)/s\phi_2 + (l-1)/ls\phi_4\}\sigma^2 \quad \text{when the } i\text{th and } i'\text{th treatments are 2nd associates,}$$

$$= 2\{(s-1)/s\phi_2 + [(p-1)\phi_3 + \phi_4]/ps\phi_3\phi_4\}\sigma^2 \quad \text{when the } i\text{th and } i'\text{th treatments are 3rd associates,}$$

$$= 2\{p/v\phi_1 + (s-1)/s\phi_2 + [l\phi_4 + \phi_3[l(p-1) - p]]/v\phi_3\phi_4\}\sigma^2 \quad \text{otherwise.}$$

Hence the average variance of all elementary treatment contrasts is given by

(4.6) $$\text{A.V.} = 2\sigma^2(v-1)^{-1}\{[(p-1)\phi_2 + lp(s-1)\phi_1]/\phi_1\phi_2 + (l-1)[(p-1)\phi_3 + \phi_4]/\phi_3\phi_4\},$$

and the efficiency of the GRAD will be

(4.7) $$(v-1)\{[(p-1)\phi_2 + lp(s-1)\phi_1]/\phi_1\phi_2 + (l-1)[(p-1)\phi_3 + \phi_4]/\phi_3\phi_4\}^{-1}r^{-1}.$$

**5. Construction of a balanced asymmetrical factorial design in $vs^{m-1}$ plots, where $s$ is a prime or a prime power.** Let $A$ be a factor at $v$ levels and $B_1, B_2, \cdots, B_m$ be $m$ factors each at $s$ levels, where $s$ is a prime or a prime power. By choosing a suitable interaction $Z$, we can form $s$ fractions of the $s^m$ treatment combinations involving the factors $B_1, B_2, \cdots, B_m$ based on the identity relationship $I = Z$, and let the $i$th fraction be denoted by $X_i$ $(i = 0, 1, 2, \cdots, s-1)$. Choose a BIB design with $v$ treatments and let the remaining parameters of this BIB design be $b$, $r$, $k$ and $\lambda$. We identify these treatments with the $v$ levels of the factor $A$. Choose the $j$th block of the design and with each level of the factor $A$ occurring in that block associate all possible levels of the factors $B_1, B_2, \cdots, B_m$ given by the fraction $X_i$. Similarly, consider the complementary of the $j$th block and with each level of the factor $A$ occurring in it associate all possible levels of the factors $B_1, B_2, \cdots, B_m$ given by the fraction $X_i'$. Include all these $vs^{m-1}$ treatment combinations into a block and designate it by $\xi_{j,i,i'}$. Then it can be seen that the $bs(s-1)$ blocks of the form $\xi_{j,i,i'}$ $(j = 1, 2, \cdots, b;\ i \neq i' = 0, 1, 2, \cdots, s-1)$ form an [13] $(s-1)$-resolvable GRAD with parameters

(5.1) $$v^* = vs^m,\quad b^* = s(s-1)b,\quad r^* = b(s-1),\quad k^* = vs^{m-1},\quad \lambda_1 = r^*,$$
$$\lambda_2 = 0,\quad \lambda_3 = (s-1)[b - 2(r-\lambda)],\quad \lambda_4 = 2(r-\lambda),$$

where $j$th replication consists of $s(s-1)$ blocks.

It can be observed that the confounded interactions in above design would be $Z$ and $AZ$. The coefficients of the treatment combinations in these interactions, when written in the form of a column vector, with the treatment combinations suitably numbered, can be seen to be the complete set of characteristic vectors corresponding to the roots $\phi_1$ and $\phi_4$ of the $C$-matrix of GRAD. Let $l_{1i_1}$ and $l_{4i_2}$ denote $\{i_1 = 1, 2, \cdots, s-1;\ i_2 = 1, 2, \cdots, (s-1)(v-1)\}$ the $(s-1)$ and $(v-1)(s-1)$ normalised orthogonal characteristic vectors corresponding to roots $\phi_1$ and $\phi_4$. Then the estimates and the error variances of the interactions belonging to each df of interactions $Z$ and $AZ$ would be given by

$$l_{1i_1}'Q/\phi_1,\ l_{4i_2}'Q/\phi_4 \quad \text{and} \quad \sigma^2/\phi_1,\ \sigma^2/\phi_4$$

respectively. In an unconfounded experiment, the error variances of these interactions would be $\sigma^2/r^*$, so that the relative loss of information on each of $(s-1)$ confounded df of $Z$ is given by $(r^* - \phi_1)/r^*$ and on each of $(v-1)(s-1)$ confounded df of $AZ$ is given by $(r^* - \phi_4)/r^*$. Thus the total loss of information on both the confounded interactions $Z$ and $AZ$ would be

TABLE 5.1
*Plan of* $5 \times 2^2$ *factorial design*

| Replications | 1 | | 2 | | 3 | | 4 | | 5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| Blocks | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| Treat-ments | $0X_1$ | $0X_0$ | $1X_1$ | $1X_0$ | $2X_1$ | $2X_0$ | $3X_1$ | $3X_0$ | $4X_1$ | $4X_0$ |
| | $1X_1$ | $1X_0$ | $2X_1$ | $2X_0$ | $3X_1$ | $3X_0$ | $4X_1$ | $4X_0$ | $0X_1$ | $0X_0$ |
| | $2X_1$ | $2X_0$ | $3X_1$ | $3X_0$ | $4X_1$ | $4X_0$ | $0X_1$ | $0X_0$ | $1X_1$ | $1X_0$ |
| | $3X_1$ | $3X_0$ | $4X_1$ | $4X_0$ | $0X_1$ | $0X_0$ | $1X_1$ | $1X_0$ | $2X_1$ | $2X_0$ |
| | $4X_0$ | $4X_1$ | $0X_0$ | $0X_1$ | $1X_0$ | $1X_1$ | $2X_0$ | $2X_1$ | $3X_0$ | $3X_1$ |

$$\{(r^* - \phi_1)(s-1)/r^* + (r^* - \phi_4)(s-1)(v-1)/r^*\} = s - 1, \quad = (b^*/r^*) - 1. \tag{5.2}$$

Hence the asymmetrical design with parameters (5.1) is a balanced design in $b(s-1)$ replications.

In [11] Shah had given a balanced $5 \times 2^2$ factorial design in 10 blocks of 10 plots each, which was not a resolvable design. Now by the above method we can get a balanced resolvable $5 \times 2^2$ factorial design as follows:

Let $A$ be a factor at 5 levels and $B$ and $C$ be 2 factors each at two levels. Choosing interaction $BC$, form 2 fractions of the $2^2$ treatment combinations of factors $B$ and $C$, based on the identity relationship $I = BC$. Denote the fractions (00, 11) and (10, 01) by $X_0$ and $X_1$. Consider a BIB design with parameters (5, 5, 4, 4, 3) and identify its 5 treatments with the 5 levels of factor $A$. Following the above procedure, we get the plan of a balanced resolvable $5 \times 2^2$ factorial design, which is given in Table 5.1.

Here the first symbol of a treatment combination denotes the level of the factor $A$. It can be seen that the loss of information on interaction $BC$ is 9/25 and loss of information on each df of interaction $ABC$ is 4/25. Hence in case of above design, we note that the loss of information on confounded two factor interaction is more than that given by Shah's design, while the loss of information on three factor interaction is less than that given by his design.

**6. Some methods of constructing generalized right angular designs.** In this section, we state a theorem similar to the Theorem 2.1 of [6] and Subsections 6.2, 6.3 and 6.4 deal with the construction methods of GRADS from particular class of existing GD designs.

Following the method of Theorem 2.1 of [6] we can prove Theorem 6.1. If $N^*$, $N'$ are the incidence matrices of singular GD design and GD design with respective parameters $v^* = ps$, $m^* = p$, $n^* = s$, $b^*$, $r^* = \lambda_1^*$, $k^*$, $\lambda_1^*$, $\lambda_2^*$ and $v' = ps$, $m' = p$, $n' = s$, $b'$, $r'$, $k' = k''k^*$, $\lambda_1'$, $\lambda_2'$ and $N''$ is the incidence matrix of a BIB design with parameters $v'' = l$, $b''$, $r''$, $k''$, $\lambda''$, the treatments can be so arranged that

$$N = [N'' \times N^* \mid I_l \times N'] \tag{6.1}$$

s an incidence matrix of a GRAD with parameters

$$(6.2)\quad v = lps,\quad b = b''b^* + lb',\quad r = r''r^* + r',\quad k = k''k^*,\quad \lambda_1 = r''r^* + \lambda_1',$$
$$\lambda_2 = r''\lambda_2^* + \lambda_2',\quad \lambda_3 = r^*\lambda'',\quad \lambda_4 = \lambda''\lambda_2^*.$$

ILLUSTRATION 6.1.1. Consider singular GD design and semiregular GD design with respective parameters $v^* = 9 = b^*, r^* = k^* = \lambda_1^* = 3, \lambda_2^* = 0$ and $v' = 9 = b', r' = 6 = k', \lambda_1' = 3, \lambda_2' = 4$ and BIB design with parameters $v'' = 3 = b''$, $r'' = 2 = k''$, $\lambda'' = 1$. From Theorem 6.1., we get the incidence matrix $N$ of a GRAD with parameters

$$(6.3)\quad v = 27,\quad p = l = s = 3,\quad b = 54,\quad r = 12,\quad k = 6,\quad \lambda_1 = 9,\quad \lambda_2 = 4,$$
$$\lambda_3 = 3,\quad \lambda_4 = 0.$$

The efficiency of the above design can be seen, using (4.7) to be 0.7956. The efficiency of the usual 3-dimensional lattice for testing $3^3$ experiment in blocks of 3 plots can be seen to be 0.591. Thus the design proposed in the above illustration is more efficient than the usual 3-dimensional lattice design.

6.2. For GD design with $v = mn = pls, m = p, n = ls$, the corresponding GD association scheme can be arranged to form the rectangular association scheme of GRAD with $v = pls$ treatments as follows:

Consider the groups of the GD association scheme as $p$ rows each of $ls$ treatments. Divide each row of $ls$ treatments to form $l$ subgroups each of $s$ treatments. It can be seen that this arrangement is nothing but the rectangular array of the association scheme of GRAD with $v = pls$ treatments.

6.2a. If a GD design exists with parameters

$$(6.4)\quad v' = mn,\quad m = p,\quad n = ls,\quad b' = lr',\quad r', k' = sp,\quad \lambda_1', \lambda_2',$$

then a GRAD with parameters

$$(6.5)\quad v = pls,\quad b = b' + l,\quad r = r' + 1,\quad k = k',\quad \lambda_1 = \lambda_1' + 1,\quad \lambda_2 = \lambda_2' + 1,$$
$$\lambda_3 = \lambda_1',\quad \lambda_4 = \lambda_2'$$

can be constructed as follows:

Write down the GD association scheme of (6.4) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design (6.4) we add $l$ blocks such that the $g$th block consists of $sp$ treatments from $p$ subgroups denoted by $(1, g), (2, g), \cdots, (p, g)$ of the so formed association scheme of GRAD $\{f = 1, 2, \cdots, p; g = 1, 2, \cdots, l\}$. Now it can be verified that these $b' + l$ blocks form a GRAD with parameters (6.5).

ILLUSTRATION 6.2a.1. Following the above method, we can transform the GD design SR 25 of the PBIB design tables [2], with parameters

$$(6.6)\quad v' = 12 = b',\quad m = 3,\quad n = 4,\quad r' = 6 = k',\quad \lambda_1' = 2,\quad \lambda_2' = 3$$

into a GRAD with parameters

$$(6.7)\quad v = 12,\quad p = 3,\quad l = 2 = s,\quad b = 14,\quad r = 7,\quad k = 6,\quad \lambda_1 = 3,$$
$$\lambda_2 = 4,\quad \lambda_3 = 2,\quad \lambda_4 = 3.$$

6.2b. If a GD design exists with parameters

$$(6.8)\quad v' = mn,\quad m = p,\quad n = ls,\quad b' = pr',\quad r', k' = ls,\quad \lambda_1', \lambda_2',$$

then a GRAD with parameters

$$v = pls,\quad b = b' + p(p-1)l,\quad r = r' + l(p-1),\quad k = k',$$

$$(6.9)\quad \lambda_1 = \lambda_1' + l(p-1),\quad \lambda_2 = \lambda_2',\quad \lambda_3 = \lambda_1' + (l-2)(p-1),$$
$$\lambda_4 = \lambda_2' + 2$$

can be constructed as follows:

Write down the GD association scheme of (6.8) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design (6.8) we add $p(p-1)l$ blocks such that $ff'g$th block consists of $ls$ treatments from the subgroups $(f, g)$, $(f', 1)$, $(f', 2)$, $\cdots$, $(f', g-1)$, $(f', g+1)$, $\cdots$, $(f', l)$ of the so formed association scheme of GRAD $\{f \neq f' = 1, 2, \cdots, p;\ g = 1, 2, \cdots, l\}$. Now it can be verified that these $b' + p(p-1)l$ blocks form a GRAD with parameters (6.9).

ILLUSTRATION 6.2b.1. Following the above method, we can transform the GD design SR 40 of the PBIB design tables [2], with parameters

$$(6.10)\quad v' = 16 = b',\quad m = 4 = n,\quad r' = 4 = k',\quad \lambda_1' = 0,\quad \lambda_2' = 1.$$

into a GRAD with parameters

$$(6.11)\quad v = 16,\quad p = 4,\quad l = 2 = s,\quad b = 40,\quad r = 10,\quad k = 4,$$
$$\lambda_1 = 6,\quad \lambda_2 = 1,\quad \lambda_3 = 0,\quad \lambda_4 = 3.$$

6.2c. If a GD design exists with parameters

$$(6.12)\quad v' = mn,\quad m = p,\quad n = ls,\quad b', r', k' = 2s,\quad \lambda_1', \lambda_2', l, p > 2,$$

then a GRAD with parameters

$$(6.13)\quad v = pls,\quad b = b' + \tfrac{1}{2}p(p-1)l,\quad r = r' + (p-1),\quad k = k',$$
$$\lambda_1 = \lambda_1' + (p-1),\quad \lambda_2 = \lambda_2' + 1,\quad \lambda_3 = \lambda_1',\quad \lambda_4 = \lambda_2'.$$

can be constructed as follows:

Write down the GD association scheme of (6.12) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design with parameters (6.12) we add $\frac{1}{2}p(p-1)l$ blocks formed as below. For every $(\alpha, \beta)$th subgroup of the so formed association scheme of GRAD, we form blocks consisting of $2s$ treatments where the $f$th block $(f = 1, 2, \cdots, p; f > \alpha)$ consists of the treatments from the subgroups $(\alpha, \beta)$ and $(f, \beta)$. Now it can be verified that these $b' + \frac{1}{2}p(p-1)l$ blocks form a GRAD with parameters (6.13).

ILLUSTRATION 6.2c.1. Following the above method, we can transform the GD design SR 84 of the PBIB design tables [2], with parameters

$$(6.14)\quad v' = 54,\quad m = 6,\quad n = 9,\quad b' = 81,\quad r' = 9,$$
$$k' = 6,\quad \lambda_1' = 0,\quad \lambda_2' = 1$$

into a GRAD with parameters

$$(6.15)\quad v = 54, \quad p = 6, \quad l = 3 = s, \quad b = 126, \quad r = 14, \quad k = 6,$$
$$\lambda_1 = 5, \quad \lambda_2 = 2, \lambda_3 = 0, \quad \lambda_4 = 1.$$

6.2d. If a GD design exists with parameters

$$(6.16)\quad v' = mn, \quad m = p, \quad n = ls, \quad b', r', k' = s + 1, \quad \lambda_1', \lambda_2',$$

then a GRAD with parameters

$$(6.17)\quad v = pls, \quad b = b' + v(p-1), \quad r = r' + (s+1)(p-1), \quad k = k',$$
$$\lambda_1 = \lambda_1' + s(p-1), \quad \lambda_2 = \lambda_2' + 2, \quad \lambda_3 = \lambda_1', \quad \lambda_4 = \lambda_2'$$

can be constructed as follows:

Write down the GD association scheme of (6.16) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design with parameters (6.16) we add $p(p-1)ls$ blocks such that $ff'gh$th block consists of $s$ treatments of the $(f, g)$th subgroup and the treatment $(f', g, h)$ of the so formed association scheme of GRAD $\{f \neq f' = 1, 2, \cdots, p; g = 1, 2, \cdots, l; h = 1, 2, \cdots, s\}$. Now it can be verified that these $b' + p(p-1)ls$ blocks form a GRAD with parameters (6.17).

Illustration 6.2d.1. Following the above method, we can transform the GD design R 35, of the PBIB design tables [2], with parameters

$$(6.18)\quad v' = 16, \quad m = 4 = n, \quad b' = 32, \quad r' = 6, \quad k' = 3, \quad \lambda_1' = 0, \quad \lambda_2' = 1$$

into a GRAD with parameters

$$(6.19)\quad v = 16, \quad p = 4, \quad l = 2 = s, \quad b = 80, \quad r = 15, \quad k = 3,$$
$$\lambda_1 = 6, \quad \lambda_2 = 3, \quad \lambda_3 = 0, \quad \lambda_4 = 1.$$

6.3. For GD designs with $v' = mn = pls, m = l, n = ps$ the corresponding GD association scheme can be arranged to form the rectangular association scheme of GRAD with $v = pls$ treatments as follows:

Consider the $l$ groups of the GD association scheme as $l$ columns of $ps$ treatments. Divide each row of $ps$ treatments to form $p$ subgroups each of $s$ treatments. It can be seen that this arrangement is nothing but the rectangular array of the association scheme of GRAD with $v = pls$ treatments.

6.3a. If a GD design exists with parameters

$$(6.20)\quad v' = mn, \quad m = l, \quad n = ps, \quad b' = lr', \quad r', k' = ps, \quad \lambda_1', \lambda_2',$$

then a GRAD with parameters

$$v = pls, \quad b = b' + pl(l-1), \quad r = r' + p(l-1), \quad k = k',$$
$$(6.21)\quad \lambda_1 = \lambda_1' + p(l-1), \quad \lambda_2 = \lambda_1' + (p-2)(l-1),$$
$$\lambda_3 = \lambda_2', \quad \lambda_4 = \lambda_2' + 2$$

can be constructed as follows:

Write down the GD association scheme of (6.20) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design (6.20) we add $pl(l-1)$ blocks such that $fgg'$th block consists of $sp$ treatments from the subgroups $(f, g)$, $(1, g')$, $(2, g')$, $\cdots$, $(f-1, g')$, $(f+1, g')$, $\cdots$, $(p, g')$ of the so formed association scheme of GRAD $\{f = 1, 2, \cdots, p; g \neq g' = 1, 2, \cdots, l\}$. Now it can be verified that these $b' + pl(l-1)$ blocks form a GRAD with parameters (6.21).

ILLUSTRATION 6.3a.1. Following the above method, we can transform the GD design SR 31 of the PBIB design tables [2], with parameters

$$(6.22)\quad v' = 12,\quad m = 2,\quad n = 6,\quad b' = 20,\quad r' = 10,\quad k' = 6,\quad \lambda_1' = 4,\quad \lambda_2' = 5,$$

into a GRAD with parameters

$$(6.23)\quad v = 12,\quad p = 3,\quad l = 2 = s,\quad b = 26,\quad r = 13,\quad k = 6,\quad \lambda_1 = 7 = \lambda_4,\quad \lambda_2 = 5 = \lambda_3.$$

6.3b. If a GD design exists with parameters

$$(6.24)\quad v' = mn,\quad m = l,\quad n = ps,\quad b', r', k' = 2s,\quad \lambda_1', \lambda_2', l, p > 2,$$

then a GRAD with parameters

$$(6.25)\quad v = pls,\quad b = b' + \tfrac{1}{2}pl(l-1),\quad r = r' + (l-1),\quad k = k',\quad \lambda_1 = \lambda_1' + (l-1),\quad \lambda_2 = \lambda_1',\quad \lambda_3 = \lambda_2' + 1,\quad \lambda_4 = \lambda_2'$$

can be constructed as follows:

Write down the GD association scheme of (6.24) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design with parameters (6.24) we add $\frac{1}{2}pl(l-1)$ blocks formed as below. For every $(\alpha, \beta)$th subgroup of the so formed association scheme of GRAD, we form blocks consisting of $2s$ treatments where the $g$th block $(g = 1, 2, \cdots, l; g > \beta)$ consists of the treatments from the subgroups $(\alpha, \beta)$ and $(\alpha, g)$. Now it can be verified that these $b' + \frac{1}{2}pl(l-1)$ blocks form a GRAD with parameters (6.25).

ILLUSTRATION 6.3b.1. Following the above method, we can transform the GD design R 49 of the PBIB design tables [2], with parameters

$$(6.26)\quad v' = 24,\quad m = 3,\quad n = 8,\quad b' = 60, r' = 10,\quad k' = 4,\quad \lambda_1' = 2,\quad \lambda_2' = 1$$

into a GRAD with parameters

$$(6.27)\quad v = 24,\quad p = 4,\quad l = 3,\quad s = 2,\quad b = 72,\quad r = 12,\quad k = 4,\quad \lambda_1 = 4,\quad \lambda_2 = 2 = \lambda_3,\quad \lambda_4 = 1.$$

6.3c. If a GD design exists with parameters

$$(6.28)\quad v' = mn,\quad m = l,\quad n = ps,\quad b', r', k' = s + 1,\quad \lambda_1', \lambda_2',$$

then a GRAD with parameters

$$v = pls, \quad b = b' + v(l-1), \quad r = r' + (s+1)(l-1), \quad k = k', \tag{6.29}$$
$$\lambda_1 = \lambda_1' + s(l-1), \quad \lambda_2 = \lambda_1', \quad \lambda_3 = \lambda_2' + 2, \quad \lambda_4 = \lambda_2'$$

can be constructed as follows:

Write down the GD association scheme of (6.28) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design with parameters (6.28) we add $pl(l-1)s$ blocks such that $fgg'h$th block consists of $s$ treatments of the $(f, g)$th subgroup and the treatment $(f, g', h)$ of the so formed association scheme of GRAD $\{f = 1, 2, \cdots, p; g \neq g' = 1, 2, \cdots, l; h = 1, 2, \cdots, s\}$.

Now it can be verified that these $b' + pl(l-1)s$ blocks form a GRAD with parameters (6.29).

ILLUSTRATION 6.3c.1. Following the above method, we can transform the GD design, SR 45 of the PBIB design tables [2], with parameters

$$v' = 18, \quad m = 3, \quad n = 6, \quad b' = 36, \tag{6.30}$$
$$r' = 6, \quad k' = 3, \quad \lambda_1' = 0, \quad \lambda_2' = 1$$

into a GRAD with parameters

$$v = 18, \quad p = 3, \quad l = 3, \quad s = 2, \quad b = 72, \quad r = 12, \quad k = 3, \tag{6.31}$$
$$\lambda_1 = 4, \quad \lambda_2 = 0, \quad \lambda_3 = 3, \quad \lambda_4 = 1.$$

6.4. For GD designs with $v' = mn = pls$, $m = pl$, $n = s$ the corresponding GD association scheme can be arranged to form the rectangular association scheme of GRAD with $v = pls$ treatments as follows:

Consider the $pl$ groups of the corresponding GD association scheme as subgroups and arrange them to form a rectangular array of subgroups of $l$ columns and $p$ rows. It can be seen that this arrangement is nothing but the rectangular array of the association scheme of GRAD with $v = pls$ treatments.

6.4a. If a GD design exists with parameters

$$v' = mn, \quad m = pl, \quad n = s, \quad b' = lr', \quad r', k' = sp, \lambda_1', \lambda_2', \tag{6.32}$$

then a GRAD with parameters

$$v = pls, \quad b = b' + l(l-1)p, \quad r = r' + p(l-1), \quad k = k',$$
$$\lambda_1 = \lambda_1' + p(l-1), \quad \lambda_2 = \lambda_2' + (p-2)(l-1), \tag{6.33}$$
$$\lambda_3 = \lambda_2', \quad \lambda_4 = \lambda_2' + 2$$

can be constructed as follows:

Write down the GD association scheme of (6.32) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design (6.32) we add $pl(l-1)$ blocks following the procedure of 6.3a. Now it can be verified that these $b' + l(l-1)p$ blocks form a GRAD with parameters (6.33).

ILLUSTRATION 6.4a.1. Following the above method, we can transform the GD design SR 55 of the PBIB design tables [2], with parameters

$$(6.34)\quad v' = 20,\quad m = 10,\quad n = 2,\quad b' = 16,\quad r' = 8,\quad k' = 10,\quad \lambda_1' = 0,\quad \lambda_2' = 4$$

into a GRAD with parameters

$$(6.35)\quad v = 20,\quad p = 5,\quad l = 2,\quad s = 2,\quad b = 26,\quad r = 13,\quad k = 10,\quad \lambda_1 = 5,\quad \lambda_2 = 7,\quad \lambda_3 = 4,\quad \lambda_4 = 6.$$

6.4b. If a GD design exists with parameters

$$(6.36)\quad v' = mn,\quad m = pl,\quad n = s,\quad b' = pr',\quad r', k' = ls, \lambda_1', \lambda_2',$$

then a GRAD with parameters

$$(6.37)\quad \begin{gathered} v = pls,\quad b = b' + lp(p - 1),\quad r = r' + l(p - 1),\quad k = k', \\ \lambda_1 = \lambda_1' + l(p - 1),\quad \lambda_2 = \lambda_2',\quad \lambda_3 = \lambda_2' + (l - 2)(p - 1), \\ \lambda_4 = \lambda_2' + 2, \end{gathered}$$

can be constructed as follows:

Write down the GD association scheme of (6.36) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design (6.36) we add $p(p - 1)l$ blocks following the procedure of 6.2b. Now it can be verified that these $b' + p(p - 1)l$ blocks form a GRAD with parameters (6.37).

ILLUSTRATION 6.4b.1. Following the above method, we can transform the GD design, SR 49 of the PBIB design tables [2], with parameters

$$(6.38)\quad v' = 18,\quad m = 6,\quad n = 3,\quad b' = 27,\quad r' = 9,\quad k' = 6,\quad \lambda_1' = 0,\quad \lambda_2' = 3$$

into a GRAD with parameters

$$(6.39)\quad v = 18,\quad p = 3,\quad l = 2,\quad s = 3,\quad b = 39,\quad r = 13,\quad k = 6,\quad \lambda_1 = 4,\quad \lambda_2 = 3 = \lambda_3,\quad \lambda_4 = 5.$$

6.4c. If a GD design exists with parameters

$$(6.40)\quad v' = mn,\quad m = lp,\quad n = s,\quad b', r', k' = 2s,\quad \lambda_1', \lambda_2', l, p > 2,$$

then a GRAD with parameters

$$(6.41)\quad \begin{gathered} v = pls,\quad b = b' + \tfrac{1}{2}lp(l + p - 2),\quad r = r' + (l + p - 2),\quad k = k', \\ \lambda_1 = \lambda_1' + (l + p - 2),\quad \lambda_2 = \lambda_2' + 1 = \lambda_3,\quad \lambda_4 = \lambda_2', \end{gathered}$$

can be constructed as follows:

Write down the GD association scheme of (6.40) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design (6.40) we add $\frac{1}{2}p(p - 1)l$ and $\frac{1}{2}pl(l - 1)$ blocks following the procedures of

6.2c and 6.3b respectively. Now it can be verified that these $b' + \frac{1}{2}lp(l + p - 2)$ blocks form a GRAD with parameters (6.41).

ILLUSTRATION 6.4c.1. Following the above method, we can transform the GD design, R 57 of the PBIB design tables [2], with parameters

$$(6.42)\quad v' = 30,\quad m = 15,\quad n = 2,\quad b' = 75,\quad r' = 10,\quad k' = 4,\quad \lambda_1' = 2,\quad \lambda_2' = 1$$

into a GRAD with parameters

$$(6.43)\quad v = 30,\quad p = 5,\quad l = 3,\quad s = 2,\quad b = 120,\quad r = 16,\quad k = 4,\quad \lambda_1 = 8,\quad \lambda_2 = \lambda_3 = 2,\quad \lambda_4 = 1.$$

6.4d. If a GD design exists with parameters

$$(6.44)\quad v' = mn,\quad m = pl,\quad n = s,\quad b', r', k' = s + 1,\quad \lambda_1', \lambda_2',$$

then a GRAD with parameters

$$(6.45)\quad v = pls,\quad b = b' + v(l + p - 2),\quad r = r' + (s + 1)(l + p - 2),$$
$$k = k',\quad \lambda_1 = \lambda_1' + s(l + p - 2),\quad \lambda_2 = \lambda_2' + 2 = \lambda_3,\quad \lambda_4 = \lambda_2',$$

can be constructed as follows:

Write down the GD association scheme of (6.44) in the form of the association scheme of GRAD with $v = pls$ treatments. To the $b'$ blocks of the GD design (6.44) we add $v(p - 1)$ and $v(l - 1)$ blocks following the procedures of 6.2d and 6.3c respectively. Now it can be verified that these $b' + v(l + p - 2)$ blocks form a GRAD with parameters (6.45).

ILLUSTRATION 6.4d.1. Following the above method, we can transform the GD design, R 16 of the PBIB design tables [2], with parameters

$$(6.46)\quad v' = 12,\quad m = 6,\quad n = 2,\quad b' = 20,\quad r' = 5,\quad k' = 3,\quad \lambda_1' = 0,\quad \lambda_2' = 1,$$

into a GRAD with parameters

$$v = 12,\quad p = 3,\quad l = 2 = s,\quad b = 56,\quad r = 14,\quad k = 3,\quad \lambda_1 = 6,\quad \lambda_2 = 3 = \lambda_3,\quad \lambda_4 = 1.$$

**7. Some combinatorial properties of generalized right angular designs.** From the association scheme of GRAD it follows that

THEOREM 7.1. *If in a generalized right angular design,* (i) $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$ *or* (ii) $\lambda_1 = \lambda_3 \neq \lambda_2 = \lambda_4$ *or* (iii) $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$ *then the design reduces to group divisible design and if* (i) $\lambda_1 \neq \lambda_2$ *and* $\lambda_2 \neq \lambda_3 = \lambda_4$ *or* (ii) $\lambda_1 \neq \lambda_3$ *and* $\lambda_2 = \lambda_4$ *then the design reduces to* GD 3*-associate design.*

Denote the sets $(\alpha, \beta) \cup \{\bigcup_{\alpha'}(\alpha', \beta')\}$ and $(\alpha, \beta) \cup \{\bigcup_{\beta'}(\alpha', \beta')\}$ respectively by $E_1$ and $E_2$ $\{\alpha \neq \alpha' = 1, 2, \cdots, p; \beta \neq \beta' = 1, 2, \cdots, l\}$, where $\cup$ denotes the set union and $\bigcup_{\alpha' (\text{or } \beta')}(\alpha', \beta')$ denotes the union of $(\alpha', \beta')$ over $\alpha'$ (or $\beta'$).

Following the method of Theorem 6.2 of [16] we can prove the

THEOREM 7.2. *In a generalized right angular design*

(i) *if* $\theta_1 = 0$, *then* $k/p$ *is an integer and every block contains* $k/p$ *treatments from subgroups having the same position in different groups.*

(ii) *If* $\theta_3 = 0$, *then* $k/l$ *is an integer and every block contains* $k/l$ *treatments from each of the* $l$ *groups.*

(iii) *If* $\theta_3 = \theta_4 = 0$, *then* $k/l$ *is an integer and every block contains* $k/l$ *treatments from the set* $E_1$ *and* $\lambda_2$ *equals* $\lambda_4$.

(iv) *If* $\theta_1 = \theta_4 = 0$, *then* $k/p$ *is an integer and every block contains* $k/p$ *treatments from the set* $E_2$ *and the design reduces to* GD 3-*associate design.*

(v) *If* $\theta_1 = \theta_3 = \theta_4 = 0$, *then* $k/pl$ *is an integer and every block contains* $k/pl$ *treatments from each of* $pl$ *subgroups of the association scheme and the design reduces to group divisible design.*

The designs with the following parameters violate the conditions of the above theorem and hence are nonexistent. The reason for nonexistence is shown in parenthesis against the parameters.

| $v$ | $b$ | $p$ | $l$ | $s$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 18 | 27 | 3 | 3 | 2 | 12 | 8 | 8 | 3 | 6 | 5 | ($\theta_3 = 0$, $k/l \neq$ integer) |
| 24 | 16 | 4 | 2 | 3 | 12 | 18 | 12 | 10 | 6 | 8 | ($\theta_1 = 0$, $k/p \neq$ integer) |
| 24 | 16 | 4 | 2 | 3 | 12 | 18 | 9 | 10 | 8 | 8 | ($\theta_1 = \theta_4 = 0$, $k/p \neq$ integer). |

Other necessary conditions for the existence of a GRAD with one of the characteristic roots zero can be obtained by the application of a theorem proved in [14]. For a brief resume of the properties of the Legendre symbol, Hilbert norm residue symbol and the Hasse-Minkowski invariant, we refer to [14]. In case of designs having generalized right angular association scheme, it can be seen that the sets of orthogonal, rational, characteristic vectors corresponding to the roots $\theta_0$, $\theta_1$, $\theta_2$, $\theta_3$ and $\theta_4$ respectively are the column vectors of the five matrices

$$E_{v,1};\ E_{l,1} \times \begin{bmatrix} 1 & 1 & & 1 \\ -1 & 1 & & 1 \\ 0 & -2 & & \cdot \\ \cdot & 0 & & \cdot \\ 0 & 0 & & -(p-1) \end{bmatrix} \times E_{s,1};\quad I_{pl} \times \begin{bmatrix} 1 & 1 & & 1 \\ -1 & 1 & & 1 \\ 0 & -2 & & \cdot \\ \cdot & 0 & & \cdot \\ 0 & 0 & & -(s-1) \end{bmatrix};$$

$$\begin{bmatrix} 1 & 1 & & 1 \\ -1 & 1 & & 1 \\ 0 & -2 & & \cdot \\ \cdot & 0 & & \cdot \\ 0 & 0 & & -(l-1) \end{bmatrix} \times E_{sp,1};\quad \begin{bmatrix} 1 & 1 & & 1 \\ -1 & 1 & & 1 \\ 0 & -2 & & \cdot \\ \cdot & 0 & & \cdot \\ 0 & 0 & & -(l-1) \end{bmatrix} \tag{7.1}$$

$$\times \begin{bmatrix} 1 & 1 & & 1 \\ -1 & 1 & & 1 \\ 0 & -2 & & \cdot \\ \cdot & 0 & & \cdot \\ 0 & 0 & & -(p-1) \end{bmatrix} \times E_{s,1}.$$

Let $Q_0$, $Q_1$, $Q_2$, $Q_3$ and $Q_4$ denote the gramians corresponding to the column vectors of the above given five matrices. The values and Hasse-Minkowski invariants of the gramians $Q_0$, $Q_1$, $Q_2$, $Q_3$ and $Q_4$ can be evaluated as

$$|Q_0| = v; \quad |Q_1| = ls^{p-1}\textstyle\prod_{j=1}^{p-1} j(j+1); \quad |Q_2| = \{\prod_{j=1}^{s-1} j(j+1)\}^{pl};$$

$$|Q_3| = \textstyle\prod_{j=1}^{l-1} j(j+1)(sp)^{l-1}; \quad |Q_4| = \prod_{j=1}^{l-1} j(j+1)\{\prod_{j=1}^{p-1} j(j+1)s^{p-1}\}^{l-1};$$

$$\begin{aligned} C_p(Q_0) &= (-1, v)_p; \\ C_p(Q_1) &= (-1,-1)_p(s,p)^{p-2}\cdot(ls,-1)^{\frac{1}{2}p(p-1)}\cdot(s^{p-1}p, l)^{p-2}; \\ C_p(Q_2) &= (-1,-1)_p(s,-1)^{\frac{1}{2}lp(lp+1)}; \qquad (7.2) \\ C_p(Q_3) &= (-1,-1)_p(sp,l)^{l-2}\cdot(sp,-1)^{\frac{1}{2}l(l-1)}; \\ C_p(Q_4) &= (-1,-1)_p(s,-1)^{\frac{1}{2}p(p-1)(l-1)}\cdot(s,p)^{(l-1)(p-2)}\cdot(l,ps^{p-1})^{(l-1)(p-1)-1} \\ &\quad \cdot(l,-1)^{\frac{1}{2}(p-1)(p-2)}\cdot(ps^{p-1},-1)^{\frac{1}{2}(l-1)(l-2)}. \end{aligned}$$

By application of Theorem 3.2 of [14] and necessary simplifications we have the following theorems.

THEOREM 7.3. *Necessary conditions for the existence of a generalized right angular design with* $\theta_1 = 0$ *and* $b = p(sl-1)+1$ *are*

$$b\cdot s^{p-2}\cdot\theta_2^{pl(s-1)}\cdot\theta_3^{l-1}\cdot\theta_4^{(l-1)(p-1)} \sim 1, \qquad (7.3)$$

*and further, if* (7.3) *is satisfied then,*

$$\begin{aligned} &(\theta_0, -v)_p(v, \textstyle\prod_{i=2}^4 \theta_i^{\alpha_i})_p\{\prod_{i=2}^4 (-1,\theta_i)_p^{\frac{1}{2}\alpha_i(\alpha_i+3)}\}\cdot\{\prod_{i<j=2;\,i,j\neq 1}^4(\theta_i^{\alpha_i},\theta_j^{\alpha_j})_p\} \\ &\cdot\{\textstyle\prod_{i<j=2;\,i,j\neq 1}^4(\theta_i^{\alpha_i}, |Q_j|)_p\}\{\prod_{i<j=2;\,i,j\neq 1}^4(\theta_j^{\alpha_j}, |Q_i|)_p\} \qquad (7.4) \\ &\cdot\{\textstyle\prod_{i=2}^4(\theta_i, |Q_i|)_p^{\alpha_i-1}\}\cdot C_p(Q_1) = 1. \end{aligned}$$

THEOREM 7.4. *Necessary conditions for the existence of a generalized right angular design with* $\theta_2 = 0$ *and* $b = pl$ are

$$\theta_0\cdot\theta_1^{p-1}\cdot\theta_3^{l-1}\cdot\theta_4^{(l-1)(p-1)}\cdot s^{pl} \sim 1, \qquad (7.5)$$

*and further, if* (7.5) *is satisfied then,*

$$\begin{aligned} &(\theta_0, -v)_p(v, \textstyle\prod_{i=1,i\neq 2}^4 \theta_i^{\alpha_i})_p\{\prod_{i=1,i\neq 2}^4 (-1,\theta_i)^{\frac{1}{2}\alpha_i(\alpha_i+3)}\} \\ &\cdot\{\textstyle\prod_{i<j=1;\,i,j\neq 2}^4(\theta_i^{\alpha_i},\theta_j^{\alpha_j})_p\}\{\prod_{i<j=1;\,i,j\neq 2}^4(\theta_i^{\alpha_i}, |Q_j|)_p\} \qquad (7.6) \\ &\cdot\{\textstyle\prod_{i<j=1;\,i,j\neq 2}^4(\theta_j^{\alpha_j}, |Q_i|)_p\}\cdot\{\prod_{i=1,i\neq 2}^4(\theta_i, |Q_i|)_p^{\alpha_i-1}\}\cdot C_p(Q_2) = 1. \end{aligned}$$

THEOREM 7.5. *Necessary conditions for the existence of a generalized right angular design with* $\theta_3 = 0$ *and* $b = l(ps-1)+1$ *are*

$$b\cdot(sp)^{l-2}\cdot\theta_1^{p-1}\cdot\theta_2^{pl(s-1)}\cdot\theta_4^{(p-1)(l-1)} \sim 1, \qquad (7.7)$$

*and further, if* (7.7) *is satisfied then,*

$$\begin{aligned} &(\theta_0, -v)_p(v, \textstyle\prod_{i=1,i\neq 3}^4 \theta_i^{\alpha_i})_p\cdot\{\prod_{i=1,i\neq 3}^4 (-1,\theta_i)^{\frac{1}{2}\alpha_i(\alpha_i+3)}\} \\ &\cdot\{\textstyle\prod_{i<j=1;\,i,j\neq 3}^4(\theta_i^{\alpha_i},\theta_j^{\alpha_j})_p\}\cdot\{\prod_{i<j=1;\,i,j\neq 3}^4(\theta_i^{\alpha_i}, |Q_j|)_p\} \qquad (7.8) \\ &\cdot\{\textstyle\prod_{i<j=1;\,i,j\neq 3}^4(\theta_j^{\alpha_j}, |Q_i|)_p\}\cdot\{\prod_{i=1,i\neq 3}^4(\theta_i, |Q_i|)_p^{\alpha_i-1}\}\cdot C_p(Q_3) = 1. \end{aligned}$$

THEOREM 7.6. *Necessary conditions for the existence of a generalized right angular design with* $\theta_4 = 0$ *and* $b = psl - (p-1)(l-1)$ *are*

$$b \cdot p^{l-2} \cdot s^{(l-1)(p-1)-1} \cdot \theta_1^{p-1} \cdot \theta_2^{pl(s-1)} \cdot \theta_3^{l-1} \sim 1, \tag{7.9}$$

*and further, if* (7.9) *is satisfied then,*

$$\begin{aligned} &(\theta_0, -v)_p (v, \textstyle\prod_{i=1, i\neq 4}^{4} \theta_i^{\alpha_i})_p \{\prod_{i=1, i\neq 4}^{4} (-1, \theta_i)^{\frac{1}{2}\alpha_i(\alpha_i+3)}\} \\ &\cdot \{\textstyle\prod_{i<j=1; i,j\neq 4}^{4} (\theta_i^{\alpha_i}, \theta_j^{\alpha_j})_p\} \cdot \{\prod_{i<j=1; i,j\neq 4}^{4} (\theta_i^{\alpha_i}, |Q_j|)_p\} \\ &\cdot \{\textstyle\prod_{i<j=1; i,j\neq 4}^{4} (\theta_j^{\alpha_j}, |Q_i|)\} \cdot \{\prod_{i=1, i\neq 4}^{4} (\theta_i, |Q_i|)_p^{\alpha_i - 1}\} \cdot C_p(Q_4) = 1. \end{aligned} \tag{7.10}$$

The designs with the following parameters violate the conditions of the above theorems and hence are nonexistent. The necessary theorem number ruling out the existence of the designs is shown in parenthesis against the parameters

| $v$ | $b$ | $p$ | $l$ | $s$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 40 | 20 | 5 | 4 | 2 | 6 | 12 | 6 | 3 | 2 | 1 | (7.4) |
| 48 | 45 | 4 | 4 | 3 | 15 | 16 | 9 | 8 | 3 | 4 | (7.3) |
| 48 | 45 | 4 | 4 | 3 | 15 | 16 | 9 | 3 | 8 | 4 | (7.5) |
| 48 | 24 | 6 | 4 | 2 | 10 | 20 | 10 | 6 | 5 | 3 | (7.4) |
| 56 | 28 | 7 | 4 | 2 | 10 | 20 | 10 | 5 | 2 | 3 | (7.4) |
| 60 | 56 | 5 | 6 | 2 | 14 | 15 | 8 | 6 | 2 | 3 | (7.3) |
| 60 | 56 | 6 | 5 | 2 | 14 | 15 | 8 | 2 | 6 | 3 | (7.5) |
| 60 | 40 | 5 | 6 | 2 | 10 | 15 | 8 | 4 | 6 | 1 | (7.6) |

**8. Nonexistence of certain symmetrical generalized right angular designs.** We call generalized right angular designs with $\theta_i \neq 0$ $(i = 1, 2, 3, 4)$ regular generalized right angular designs. From Shrikhande's [12] and Connor and Clatworthy's [3] results, it follows that

THEOREM 8.1. *A necessary condition for the existence of a symmetrical regular generalized right angular design is* $\theta_1^{\alpha_1} \cdot \theta_2^{\alpha_2} \cdot \theta_3^{\alpha_3} \cdot \theta_4^{\alpha_4}$ *should be a perfect square.*

The following corollary is obvious.

COROLLARY 8.1.1. *The regular generalized right angular designs have*

(i) $\theta_1$ *as perfect square if* $p$ *is even and* $l$ *is odd.*

(ii) $\theta_3$ *as perfect square if* $p$ *is odd and* $l$ *is even.*

(iii) $\theta_2^{s-1}$ *as perfect square if* $l$ *and* $p$ *both are odd.*

(iv) $\theta_1\theta_3\theta_4$ *as perfect square if* $l$ *and* $p$ *both are even.*

The parameters of the following designs do not satisfy the corollary and hence are nonexistent. The reason for nonexistence is shown in parenthesis against the parameters

| $v = b$ | $p$ | $l$ | $s$ | $r = k$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | |
|---|---|---|---|---|---|---|---|---|---|
| 24 | 3 | 4 | 2 | 12 | 10 | 8 | 7 | 4 | ($\theta_3 \neq$ p.s.) |
| 36 | 6 | 3 | 2 | 9 | 8 | 2 | 6 | 1 | ($\theta_1 \neq$ p.s.) |
| 40 | 5 | 2 | 4 | 10 | 6 | 3 | 2 | 1 | ($\theta_3 \neq$ p.s.) |
| 48 | 6 | 4 | 2 | 15 | 12 | 6 | 8 | 3 | ($\theta_1\theta_3\theta_4 \neq$ p.s.) |
| 56 | 7 | 4 | 2 | 20 | 14 | 7 | 11 | 6 | ($\theta_3 \neq$ p.s.) |

We now apply the Hasse-Minkowski invariant for the above regular symmetrical designs and obtain a necessary condition for the existence of them. $C_p(NN')$ can be calculated in the usual way and after simplification, we get

$$
\begin{aligned}
C_p(NN') = {} & (-1, \theta_1)_p^{\frac{1}{2}p(p-1)} \cdot (\theta_1, |Q_1|)^{p-2} \cdot (\theta_2, -1)^{\frac{1}{2}lp(s-1)[lp(s-1)+1]} \\
& \cdot (\theta_2, |Q_2|)^{lp(s-1)-1} \cdot (-1, \theta_3)^{\frac{1}{2}l(l-1)} \cdot (\theta_3, |Q_3|)^{l-2} \\
& \cdot (-1, \theta_4)^{\frac{1}{2}(l-1)(p-1)[(l-1)(p-1)+1]} \cdot (\theta_4, |Q_4|)^{(l-1)(p-1)-1} \\
& \cdot (\theta_1^{p-1}, \theta_2^{pl(s-1)} \cdot |Q_2|)(|Q_1|, \theta_2^{pl(s-1)})(\theta_3^{l-1}, \theta_4^{(l-1)(p-1)} \cdot |Q_4|) \\
& \cdot (|Q_3|, \theta_4^{(l-1)(p-1)})(|Q_1| \cdot |Q_2|, |Q_3| \cdot |Q_4|),
\end{aligned}
\tag{8.1}
$$

for all odd primes.

THEOREM 8.2. *A necessary condition for the existence of regular symmetrical generalized right angular design is that the right hand side of* (8.1) *is equal to* $+1$, *for all odd primes.*

Considering different values of $l$ and $p$ we get

COROLLARY 8.2.1. *A necessary condition for the existence of regular symmetrical generalized right angular designs is that*

$$
\begin{aligned}
(v, \theta_1)(\theta_4, -\theta_3) &= +1, && \text{if } l \equiv 0 (\text{mod } 4) \equiv p, \\
(\theta_1\theta_4 s, lp) &= +1, && \text{if } l \equiv 0 (\text{mod } 4), p \equiv 1 (\text{mod } 4), \\
(\theta_3, -v\theta_4)(v, \theta_4) &= +1, && \text{if } l \equiv 0 (\text{mod } 4), p \equiv 2 (\text{mod } 4), \\
(-1, \theta_1\theta_4)(lp, \theta_1\theta_4 s) &= +1, && \text{if } l \equiv 0 (\text{mod } 4), p \equiv 3 (\text{mod } 4), \\
(\theta_3\theta_4, l) &= +1, && \text{if } l \equiv 1 (\text{mod } 4), p \equiv 0 (\text{mod } 4), \\
(\theta_3\theta_4, l)(\theta_1, lp) &= +1, && \text{if } l \equiv 1 (\text{mod } 4) \equiv p, \\
(-1, \theta_2^{s-1} \cdot \theta_1)(\theta_3\theta_4, l) &= +1, && \text{if } l \equiv 1 (\text{mod } 4), p \equiv 2 (\text{mod } 4), \\
(-lp, \theta_1)(\theta_3\theta_4, l) &= +1, && \text{if } l \equiv 1 (\text{mod } 4), p \equiv 3 (\text{mod } 4), \\
(-1, \theta_1)(\theta_3, v\theta_4)(v, \theta_4) &= +1, && \text{if } l \equiv 2 (\text{mod } 4), p \equiv 0 (\text{mod } 4), \\
(lp, s\theta_1\theta_4)(-1, \theta_2)^{s-1} &= +1, && \text{if } l \equiv 2 (\text{mod } 4), p \equiv 1 (\text{mod } 4), \\
(\theta_3, v\theta_4)(v, \theta_4) &= +1, && \text{if } l \equiv 2 (\text{mod } 4) \equiv p, \\
(-1, \theta_1\theta_4\theta_2^{s-1})(lp, \theta_1\theta_4 s) &= +1, && \text{if } l \equiv 2 (\text{mod } 4), p \equiv 3 (\text{mod } 4), \\
(\theta_3\theta_4, -l) &= +1, && \text{if } l \equiv 3 (\text{mod } 4), p \equiv 0 (\text{mod } 4), \\
(\theta_1, lp)(\theta_3\theta_4, l)(-1, \theta_3) &= +1, && \text{if } l \equiv 3 (\text{mod } 4), p \equiv 1 (\text{mod } 4), \\
(-1, \theta_2^{s-1} \cdot \theta_3\theta_4)(\theta_3\theta_4, l) &= +1, && \text{if } l \equiv 3 (\text{mod } 4), p \equiv 2 (\text{mod } 4), \\
(-1, \theta_1\theta_3)(l, \theta_3\theta_4) &= +1, && \text{if } l \equiv 3 (\text{mod } 4) \equiv p.
\end{aligned}
$$

The designs with following parameters violate the conditions of the above theorem and hence are nonexistent.

| $v = b$ | $p$ | $l$ | $s$ | $r = k$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ |
|---|---|---|---|---|---|---|---|---|
| 50 | 5 | 5 | 2 | 25 | 16 | 11 | 14 | 12 |
| 60 | 5 | 6 | 2 | 25 | 24 | 12 | 16 | 8 |
| 98 | 7 | 7 | 2 | 15 | 6 | 3 | 2 | 2 |

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