CYCLIC DESIGNS[1]

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1. Introduction. Cyclic designs are incomplete block designs consisting in the simplest case of a set of blocks obtained by cyclic development of an initial block. More generally, a cyclic design consists of combinations of such sets and will be said to be of size (n, k, r), where n is the number of treatments, k the block size, and r the number of replications.

It is well known (e.g. Bose and Nair [1]) that cyclic development of a suitably chosen initial block is one of the methods of generating designs with a high degree of balance in the arrangement of the treatments such as balanced incomplete block (BIB) designs and partially balanced incomplete block designs with two associate classes (PBIB(2) designs). Again, the cyclic type is a rather junior partner among the five types into which Bose and Shimamoto [2] classify PBIB(2) designs. The emphasis in these and many related papers has been understandably on the number of associate classes, the cyclic aspect being incidental. In the present article we proceed in opposite fashion putting the cyclic property first. It will be shown how cyclic designs may be systematically generated and how the non-isomorphic designs of given size may be enumerated and constructed. All such designs are PBIB designs but may have up to $\frac{1}{2}n$ associate classes. For $n \leqq 15$ and $k = 3, 4, 5$, tables of the most efficient cyclic designs are presented and comparisons with BIB and PBIB(2) designs are made.

Points which make cyclic designs attractive are:

(i) *Flexibility.* A cyclic design of size (n, k, ik) exists for all positive integers n, k, i. If n and k have a common divisor d then a "fractional set" of size $(n, k, k/d)$ exists corresponding to each d. Fractional sets may be combined with designs of size (n, k, ik) to form fresh designs, or used by themselves especially if n is large. Thus there are cyclic designs for many sizes (n, k, r) for which no PBIB(2) design is available, but the reverse may also happen.

(ii) *Ease of representation.* No plan of the experimental layout is needed since the initial block or blocks suffice.

(iii) *Youden type.* In view of their method of generation cyclic sets with $r = k$, and hence combinations of such sets, provide *automatic* elimination of heterogeneity in two directions.

(iv) *Analysis.* For cyclic designs the coefficient matrix of the normal equations is a circulix. The inverse matrix may therefore be obtained explicitly (as another circulix), thus making possible a general method of analysis. Questions of analysis

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will not be considered further here since methods given in a special case by Kempthorne [9] continue to apply with minor modifications. However, details and aids to analysis are presented in [12].

Cyclic designs as a class in their own right were introduced for $k = 2$ by Kempthorne [9] and Zoellner and Kempthorne [13]. Design aspects for the case $k = 2$, which has some special features, were considered in [6] and [7], and will not be treated in this paper. For general k cyclic designs are closely related to the circular designs of Das [5]. See also the survey of non-orthogonal designs by Pearce [11] who calls cyclic designs a "little publicized class." PBIB designs have been studied from an algebraic point of view in a series of papers by Masuyama. In some of these (e.g. [10]) reference is made to cyclic designs but no detailed results are obtained.

2. Cyclic sets. Label the treatments $0, 1, 2, \cdots, n - 1$. To fix ideas consider the arrangement of $n = 7$ treatments in blocks of size $k = 3$. The complete design of $\binom{7}{3} = 35$ distinct blocks may be set out as follows:

(1)

{012}	:	012	123	234	345	456	560	601
{013}	:	013	124	235	346	450	561	602
{014}	:	014	125	236	340	451	562	603
{015}	:	015	126	230	341	452	563	604
{024}	:	024	135	246	350	461	502	613

From any block the others in the same row may be obtained by increasing each object label in turn by 1, 2, 3, 4, 5, 6, and reducing modulo 7. The rows have been arranged to start with the block of lowest numerical value and are designated by the initial block placed in braces. We call each row a *cyclic set.*

A block may also be conveniently represented by identical beads spaced regularly on a circular necklace. Fig. 1 shows the blocks 012 and 123. The set {012} is then generated by successive unit rotations.

It is not difficult to show that each cyclic set forms a partially balanced incomplete block (PBIB) design with b (no. of blocks) $= n$ and r (no. of replications) $= k$. If object $j + i$ is an αth associate of i, so is $n - j + i$. Thus the number m of associate classes is at most $\frac{1}{2}(n - 1)$ for n odd and $\frac{1}{2}n$ for n even, but may be less, with $m = 1$ for a balanced (BIB) design. An additional feature

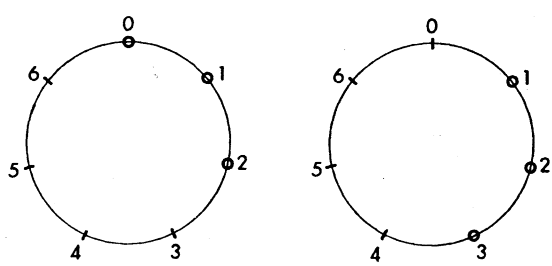

FIG. 1

of a cyclic set is that each object occurs once in each position within a block. Order effects are therefore automatically balanced out and the sets are Youden Type designs, balanced ($m = 1$) or partially balanced ($m > 1$).

The same procedure can be used for any n and k except that when n and k are not relative primes fractional sets arise consisting of n/d blocks, where d is any common divisor of n and k. In terms of Fig. 1 such sets correspond to arrangements of beads which can be reproduced in fewer than n rotations of the necklace.

For the purpose of systematically enumerating all cyclic sets it is convenient to characterize each set by a circular partition of n. Thus we may replace $\{0\, x_1\, x_2\, x_3 \cdots x_{k-2}\, x_{k-1}\}$ by $(x_1, x_2 - x_1, x_3 - x_2, \cdots, x_{k-1} - x_{k-2}, n - x_{k-1})$.

Example 1. For $n = 8$, $k = 4$ the set {0123} becomes (1115). The cyclic sets may now be written down in increasing order of the numerical value of the corresponding partition: (1115), (1124), (1133), (1142), (1214), (1223), (1232), (1313), (1322), (2222). After (1142) we omit (1151) this being identical with (1115), etc. As the repetition of digits indicates the set (1313) consists of the 4 blocks

$$0145 \quad 1256 \quad 2367 \quad 3470 \qquad (r = 2)$$

and (2222) of the 2 (disconnected) blocks 0246, 1357 ($r = 1$). These are still PBIB designs but, of course, no longer of the Youden Type. We shall say that the corresponding arrangements of beads on a necklace have *periods* 4 and 2, respectively. As a check, note that all $\binom{8}{4}$ blocks are accounted for since $8 \times 8 + 4 + 2 = 70$.

For any n and k, the total number of sets, being equal to the number of distinct arrangements of k white beads and $n - k$ black beads on a necklace of n beads (which may not be turned over) is given by (Jablonski [8])

$$(2) \qquad N(k, n - k) = n^{-1} \sum \phi(d)\{(n/d)!/[(k/d)![(n - k)/d]!]\},$$

where the summation is over all integers d (including unity) which are divisors of both k and $n - k$, and $\phi(x)$ is Euler's function, the number of positive integers less than and prime to x. Thus

$$N(4, 4) = \tfrac{1}{8}(8!/4!4! + 4!/2!2! + 2(2!/1!1!)) = 10.$$

The number of cyclic sets of various sizes making up this total is tabulated in [7] for $n \leqq 15$.

If a design of size $n = b = 7$ and $k = r = 3$ is required a look at the association schemes of the 5 sets in (1) leads to {013} or {015}, both being BIB designs. For most sizes there will be no balanced set and the choice is less clear but might be based on the usual efficiency factor. Combinations of sets provide larger designs and again the question of optimal selection of sets arises. This presents a formidable task for all but small designs. Our principal aim is to show that this task can be greatly simplified if certain isomorphisms between cyclic sets are recognized. A systematic approach for the construction of optimal cyclic designs is then developed.

3. Equivalence classes. Let us now apply to $\{012\}$ of Equation (1) the renumbering or permutation

$$R(7, 3) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 3 & 6 & 2 & 5 & 1 & 4 \end{pmatrix}$$

obtained by multiplying each of the 7 labels by 3 (mod 7). Then $\{012\}$ becomes

036 362 625 251 514 140 403,

a Youden Type design which is merely a re-arrangement of $\{014\}$. We write $\{012\} \to_3 \{014\}$. Thus $\{012\}$ and $\{014\}$ are isomorphic. Two further applications of $R(7, 3)$ give $\{024\}$ and the original $\{012\}$. We have therefore established the equivalence class $\{012\} \sim \{014\} \sim \{024\}$. No blocks need be written in the process if partition notation is used: $\{012\} \to_3 \{036\} = (331) = (133) = \{014\} \to_3 \{035\} = (322) = (223) = \{024\}$. Likewise $\{013\} \to_3 \{032\} = \{023\} = (214) = (142) = \{015\}$, so that $\{013\}$, $\{015\}$ form a second equivalence class.

The same procedure can be used for any prime n and any k. To see this, note that the permutations $R(n, 1)$ (the identity permutation), $R(n, 2), \cdots, R(n, n - 1)$, form a group under "multiplication" $*$ defined by

$$R(n, i) * R(n, j) = R(n, ij \bmod n) \tag{3}$$

which is isomorphic with the multiplicative group of residues mod n. Hence all elements $R(n, i)$ are generated by powers of $R(n, g)$, where g is a primitive root of n (i.e., $g^x \not\equiv 1 \bmod n$ for $x = 1, 2, \cdots, n - 2$ but $g^{n-1} \equiv 1 \bmod n$). But a permutation σ which changes one cyclic set into another must be of the form $R(n, i)$ if we assume without loss of generality that σ leaves 0 unchanged; for if a, b, c, d, are elements of the residue set with a and $b = a + d$ two elements in the same block we require that

$$\sigma(b) - \sigma(a) = \sigma(d) \qquad \text{all } a, b, d$$

or

$$\sigma(a) + \sigma(d) = \sigma(a + d),$$

showing that σ is multiplicative: $\sigma(a) = ca$. Thus all possible isomorphisms between cyclic sets can be established conveniently by repeated application of $R(n, g)$.

When n is not prime the $R(n, i)$ continue to form a group under $*$ of (3) provided i and j are restricted to be integers relatively prime to n. The group is now of order $\phi(n)$ and is clearly isomorphic with the multiplicative group of the reduced set of residues. g is said to be a primitive root of n if $\phi(n)$ is the smallest power making $g^{\phi(n)} \equiv 1 \bmod n$. Primitive roots exist only if n equals 2, 4, p^n, or $2p^n$, where p is any prime > 2 and n any integer. For values of n admitting a primitive root we proceed as before; otherwise, multiplication by each member of the reduced set of residues will establish most isomorphisms.

Example 1 (cont'd.) Since 8 does not have a primitive root we begin by applying $R(8, 3)$ to the sets of Example 1 and find

$$(1115) \to_3 (1232), \; (1124) \to_3 (1223), \; (1142) \to_3 (1322).$$

The other sets are unchanged by the transformation. Likewise $R(8, 5)$ gives

$$(1115) \to_5 (1232),\ (1124) \to_5 (1322),\ (1142 \to_5 (1223).$$

$R(8, 7)$ produces "mirror images" obtained by reading a circular partition anti-clockwise rather than clockwise, e.g. $(1124) \to_7 (4211) = (1142)$. This isomorphism had already been established by $R(8, 3)$ and $R(8, 5)$ because $5 \equiv -3$. However, an additional isomorphism can be obtained by the permutation

$$\binom{0}{0}\ \binom{1}{1}\ \binom{2\ 6}{6\ 2}\ \binom{3\ 7}{7\ 3}\ \binom{4}{4}\ \binom{5}{5}$$

which takes (1133) into (1214). This is the only instance we have come across where the equivalence of two cyclic sets cannot be demonstrated by a multiplicative permutation.

A listing of all equivalence classes for cyclic sets in experiments with $n \leqq 15$ and $k = 3, 4, 5$, is given in [12]. The efficiencies of these sets regarded as designs have also been tabulated. When $n = 8$, $k = 4$ we find

Design			E	E_1	E_2	E_3	E_4	
{0123}	=	(1115)	.812	.922	.834	.760	.712	
{0124}	=	(1124)	.851	.867	.873	.810	.868	
{0125}	=	(1133)	.851	.867	.809	.867	.877	
{0134}	=	(1214)	.836	.863	.810	.869	.807	
{0145}	=	(1313)	.779	.802	.803	.668	.800	$(r = 2)$

Here E is the overall efficiency and E_j $(j = 1, 2, 3, 4)$ is the efficiency factor relating to the comparison of jth associates, i.e., E_j is the ratio of $2\sigma^2/r$ to the variance of the estimated difference in treatment effects for two treatments which are jth associates. On the basis of E the choice of optimal design for $r = 4$ among the five sets (the fifth duplicated) lies between {0124} and {0125}, with the latter preferable in having only 3 associate classes. It should be noted that except for fully balanced designs the highest value of E does not necessarily

TABLE 1

Most efficient symmetric cyclic PBIB *design D for n treatments and block size k, and its efficiency E.* N.B. *Superscripts* $^{1,\,2}$ *denote respectively* BIB *and* PBIB(2) *designs*

	$k = 3$		$k = 4$		$k = 5$	
n	D	E	D	E	D	E
6	$\{013\}^2$	.784	{0123}	.895	$\{01234\}^1$	.961
7	$\{013\}^1$	.778	$\{0124\}^1$	.876	{01234}	.932
8	$\{013\}^2$	.748	{0125}	.851	$\{01235\}^2$	.914
9	{013}	.722	{0134}	.836	{01235}	.898
10	{013}	.700	{0125}	.823	{01245}	.888
11	{013}	.676	{0125}	.817	$\{01247\}^1$	.880
12	{014}	.673	$\{0137\}^2$	.813	$\{01247\}^2$	.870
13	$\{014\}^2$	.667	$\{0139\}^1$	.812	{01269}	.863
14	{014}	.670	$\{0146\}^2$	.805	{01358}	.859
15	{015}	.641	$\{0137\}^2$	.795	$\{0124\underline{10}\}$	.853

TABLE 2

Balanced (BIB) *and two-associate* PBIB *designs with codes and efficiencies from Bose* et al. [3] *and Clatworthy** [4]

	$k = 3$		$k = 4$		$k = 5$	
n	D	E	D	E	D	E
6	$R1$	.78	$S2$, $R2$	.88, .89	BIB	.96
7	BIB	.78	BIB	.88		
8	$R5$	.75	$SR7$	.84	$R108$*, $R109$*	.91, .90
9	$SR12$	.73	$R8$, $LS1$	.80, .83	$LS10$, $R112$*	.90, .89
10	$T6$	.70	$S17$, $T2$	.79, .79	$R114$*	.88
11			$T12$	.82	BIB	.88
12			$R15$	.81	$R116$*, $R117$*	.87, .87
					$R118$*	.81
13	$C1$	.67	BIB	.81		
14			$R24$	.80		
15	$T28$	.66	$R27$	.80		

correspond to the design with the smallest number of associate classes. Other optimality criteria might be used but the choice of cyclic design is in any case reduced to one of the non-isomorphic sets. Moreover, it is only combinations of these sets (and possible disconnected sets) which need to be considered in the construction of larger cyclic designs. In Table 1 we list the most efficient cyclic sets for $n \leqq 15$ and $k = 3, 4, 5$.

Cyclic sets with two associate classes. For purposes of comparison we have made a corresponding compilation in Table 2 of two-associate PBIB designs of all types as given by Bose et al. [3] and (with asterisks) by Clatworthy [4]. The BIB designs in this range are also included. It will be noted that Table 2 has gaps for several (n, k) combinations although the symmetrical case is favorable to the existence of designs with a high degree of balance. The table also shows that a cyclic design with more than two associate classes may be more efficient than any two-associate PBIB.

It is of some interest that every regular (R) group divisible PBIB of Table 2 may be laid out as a cyclic design; this is already done in [3] in some cases and may be effected for the remaining designs by suitable relabeling. We find the following isomorphisms:

$$n = 6 \ : R1 \sim \{013\}, \quad R2 \sim \{0124\};$$

$$n = 8 \ : R5 \sim \{013\}, \quad R108^* \sim \{01235\}, R109^* \sim \{01246\};$$

$$n = 9 \ : R8 \sim \{0136\}, \quad R112^* \sim \{01346\};$$

$$n = 10 : R114^* \sim \{01257\};$$

$$n = 12 : R15 \sim \{0137\}, \quad R116^* \sim \{01356\},$$
$$R117^* \sim \{01249\}, R118^* \sim \{0147\underline{10}\};$$

$$n = 14 : R24 \sim \{0146\};$$

$$n = 15 : R27 \sim \{0137\}.$$

There are only two other cyclic designs with two associate classes in the range under consideration. For $n = 13$ we have $C1 \sim \{014\}$; for $n = 12$ the design $\{01247\}$ has the same association scheme as $R116^*$ but is not isomorphic with it.

4. Combinations of cyclic sets. Cyclic sets for given n may be combined to produce a wide variety of cyclic designs, still of PBIB form. This can always be done if the number of replications r is a multiple of k but will also be possible for certain other values of r if fractional sets exist. We shall say that the combined design is of size (n, k, r). Equivalence classes may again be established. However, the most efficient cyclic design of given size is not necessarily one made up of the most efficient cyclic sets.

EXAMPLE 2. For $n = 9$, $k = 3$ we have the equivalence classes

A : (117), (225), (144);
B : (126), (243), (153), (162), (234), (135);
C : (333) $(r = 1)$.

The order within a class has been arranged so that successive sets are obtained by the application of $R(9, 2)$, the primitive root of 9 being 2. There are clearly two non-isomorphic designs of size (9, 3, 4) obtained by combining (333) with any member of Class A or Class B. Of these the latter, which may be written as $\{013, 036\}$, is the more efficient, with $E = 0.713$ and 4 associate classes.

To get designs with $r = 6$ we can take two sets from A, two from B, or one from each. Call the sets A_1, A_2, A_3, and B_1, B_2, $\cdots$, B_6. We then have the following seven equivalence classes:

$$A_1A_2,\ A_2A_3,\ A_3A_1\ ;$$
$$B_1B_2,\ B_2B_3,\ B_3B_4,\ B_4B_5,\ B_5B_6,\ B_6B_1\ ;$$
$$B_1B_3,\ B_2B_4,\ B_3B_5,\ B_4B_6,\ B_5B_1,\ B_6B_2\ ;$$
$$B_1B_4,\ B_2B_5,\ B_3B_6\ ;$$
$$A_1B_1,\ A_2B_2,\ A_3B_3,\ A_1B_4,\ A_2B_5,\ A_3B_6\ ;$$
$$A_1B_2,\ A_2B_3,\ A_3B_4,\ A_1B_5,\ A_2B_6,\ A_3B_1\ ;$$
$$A_1B_3,\ A_2B_4,\ A_3B_5,\ A_1B_6,\ A_2B_1,\ A_3B_2\ .$$

Calculations show that the most efficient cyclic design is A_1A_2 with $E = 0.731$ and 4 associate classes.

The present example has been chosen to bring out the enumeration procedure required when the original cyclic sets fall into several equivalence classes.

Actually, for $r = 6$ as many as four PBIB(2) designs are available, viz. $SR13$, $R10$, $LS3$, and $LS9^*$, of which $LS3$ is the most efficient having $E = 0.741$. When $r = 4$ the only tabulated PBIB(2) design is $LS6$, with the relatively low efficiency $E = 0.667$. For $r \leqq 10$ Table 3 lists a selection of cyclic designs in cases where no such PBIB(2) designs are known to exist or are all of more than trivially inferior efficiency.

It is of interest to note that the number of non-isomorphic designs made up

TABLE 3

Selected cyclic designs with $r > k$, corresponding optimal two-associate PBIB *designs, and efficiencies E*

Size (n, k, r)	Cyclic design	E	PBIB(2) design	E
8, 3, 6	{013, 014}	.756	$R50^*$	.747
8, 4, 5	{0134, 0246}	.850	—	
9, 3, 4	{013, 036}	.713	$LS6$	.667
10, 4, 6	{0147, 0156}	.825	$T3$	.789
10, 4, 8	{0126, 0148}	.830	$R14$	.823
11, 3, 6	{013, 026}	.727	—	
11, 3, 9	{013, 014, 027}	.730	—	
11, 4, 8	{0134, 0248}	.823	—	
13, 4, 8	{0125, 0159}	.807	$C2$	.797
13, 5, 10	{01247, 01258}	.865	—	
14, 3, 9	{014, 02$\underline{11}$, 019}	.709	—	
14, 5, 10	{0124$\underline{10}$, 017$\underline{10}$ $\underline{12}$}	.862	—	
15, 3, 4	{015, 05$\underline{10}$}	.682	$T23$	.673
15, 5, 6	{01257, 0369$\underline{12}$}	.856	$T38^*$	.808

of s sets all chosen from the same class of S sets is just $N(s, S - s)$, where N is defined by (2). This is so because we can now regard the beads of Fig. 1 as representing sets rather than blocks. The operation $R(n, g)$, where g is a primitive root, produces a unit turn. The enumeration of non-isomorphic designs when sets are from more than one class proceeds exactly as described in [7] for $k = 2$.

5. Fractional sets. The number nk of observations required for a cyclic set of size (n, k) will often be greater than desired, especially when n is large. In this situation fractional sets are very useful. As pointed out in Example 1 such sets are characterized by a repetitive pattern in their partition representation. No such design is possible if n is prime. For n composite fractional sets exist corresponding to every divisor d $(1 < d < n)$ of n since there must be at least one partition of n consisting of d repetitions. Clearly, k must be a multiple of d, and $r = k/d$; (however, $r = 1$ gives a disconnected set). From a cyclic set with parameters $(n/d, k/d)$ a fractional set with parameters $(n, k, r = k/d)$ can always be obtained.

EXAMPLE 3. For $n = 30$ connected fractional sets exist for $k = 4, 6, 8, 9, 10, \cdots$. Suppose we require a design with $k = 6$. The non-isomorphic connected cyclic sets of size (15, 3) are (11$\underline{13}$), (12$\underline{12}$), (13$\underline{11}$), (14$\underline{10}$), and (159). Of these (12$\underline{12}$) leads to the most efficient design of size (30, 6, 3), viz. (12$\underline{12}$12$\underline{12}$) or {013$\underline{15}$ $\underline{16}$ $\underline{18}$} with $E = 0.762$.

In [12] a selection of the most efficient fractional sets of given size is tabulated for $n \leqq 100$.

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