SOME INEQUALITIES FOR CENTRAL AND NON-CENTRAL DISTRIBUTIONS[1]

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1. Summary. Hájek (1962) introduced a generalized t-statistic for testing the difference of two means of normal populations having unknown, and possibly unequal variances. While the actual distribution of this statistic was unknown, he was able to obtain bounds for the type I error. The present paper is concerned with extending Hájek's result, and obtaining bounds for the power curve, as well as type I error. The result is then a test for the Behrens-Fisher problem which guarantees that the type I error will not exceed α_0, while, at the same time, the power against a specified alternative is at least β_0. Similar results are also obtained for the modified t-test, introduced by Lord (1947), in which the sample range W replaces the root-mean-square s as an estimate of standard deviation.

2. Introduction. Many testing problems in statistics lead to tests based on a statistic of the form $X/h(Z)$ where X and Z are independent random variables, with Z non-negative and $h(\cdot)$ a function whose range is a subset of the real line. Such a statistic may occur, for example, after Studentization of a random variable X. The Student t-distribution itself has this form with X distributed $N(0, 1)$, Z distributed $\chi^2(n)/n$, and $h(z) = z^{\frac{1}{2}}$.

Hájek (1962) derived some inequalities for a generalized central t-statistic in which X and $h(\cdot)$ were as above, but Z had the structure given by

$$(1) \qquad Z = \textstyle\sum_{j=1}^{k} \lambda_j \chi_j^2(m_j)/m_j, \qquad \lambda_j \geqq 0, \qquad \sum_{j=1}^{k} \lambda_j = 1,$$

where the λ_j are unknown constants, and the $\chi_j^2(m_j)/m_j$ are independent of each other, as well as of X. Hájek's main result was that the probability that $X/h(Z)$ falls into a fixed interval containing zero is less than the corresponding probability for Student's t-distribution with $m_1 + m_2 + \cdots + m_k$ degrees of freedom, but is always greater than that of the t-distribution with ν degrees of freedom where ν is any integer not exceeding $\min_{1 \leqq j \leqq k} [m_j/\lambda_j]$.

Hájek noted that such a generalized t-statistic would be useful in the Behrens-Fisher problem.

3. Remarks and notation. In the proof of the main result of Section 4 we will utilize the concavity of a special family of functions. For this reason we now introduce a concavity condition on certain random quotients.

CONDITION (A). A random quotient $X/h(Z)$ is said to satisfy condition (A)

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if, for any $a < 0$ and $b > 0$,

$$g_a(z) = P\{a < X/h(z) < 0\}$$

and

$$g_b(z) = P\{0 \leqq X/h(z) < b\}$$

are concave functions of z for $z \geqq 0$.

The introduction of the following special notation will greatly simplify the statement of the theorem of Section 4.

If $V_1, V_2, \cdots, V_m$ are independent, identically distributed random variables, then let $\bar{V}_{(r)}$ denote a random variable having the distribution of the arithmetic mean of r of these random variables. Furthermore, for any interval (a, b) containing zero, let

$$P(a, b) = P\{a < X/h(Z) < b\}$$

and

$$P_{(r)}(a, b) = P\{a < X/h(\bar{V}_{(r)}) < b\}.$$

4. Main result. The theorem which follows is a generalization of the result due to Hájek (1962). The proof, however, remains essentially the same, and is again based on Jensen's inequality which states that $Ef(X) \leqq f(EX)$ for any real-valued function f which is concave on the range of values of the random variable X.

THEOREM. *Let $\{X/h(Z): Z \in \mathcal{Z}\}$ be a collection of random quotients satisfying Condition* (A), *and let $\mathcal{Z}$ be the class of random variables having structure*

$$Z = \sum_{k=1}^{m} \lambda_k V_k, \qquad \lambda_k \geqq 0, \quad \sum_{k=1}^{m} \lambda_k = 1, \tag{2}$$

where $V_1, V_2, \cdots, V_m$ are non-negative, independent, identically distributed random variables all independent of X. Then for any interval (a, b) containing zero

$$P_{(\nu)}(a, b) \leqq P(a, b) \leqq P_{(m)}(a, b) \tag{3}$$

where ν can be any arbitrary whole number satisfying

$$\nu \leqq \min_{1 \leqq k \leqq m} [1/\lambda_k]. \tag{4}$$

PROOF. For any interval (a, b) containing zero we have

$$\begin{aligned} P(a, b) = P\{a < X/h(Z) < b\} &= EP\{a < X/h(Z) < b \mid Z\} \\ &= E[g_a(Z) + g_b(Z)] = Ef(Z) \end{aligned}$$

where f, the sum of two concave functions, is concave. In the above equation E denotes the expectation with respect to Z. We then have for any $Z \in \mathcal{Z}$,

$$P(a, b) = Ef(Z) = Ef(\textstyle\sum_{k=1}^{m} \lambda_k V_k).$$

Since the V_k are independent, identically distributed random variables, the value of $P(a, b)$ is unaltered if we replace the constants $\lambda_1, \lambda_2, \cdots, \lambda_m$ by an arbitrary permutation $\beta_1, \beta_2, \cdots, \beta_m$; that is,

$$Ef(\textstyle\sum_{k=1}^m \lambda_k V_k) = Ef(\textstyle\sum_{k=1}^m \beta_k V_k). \tag{5}$$

Utilizing (5) the remainder of the proof is exactly as given in [1], with the random variables V_k replacing the $\chi_k^2(1)$ variables of Hájek's proof.

5. Application to non-central t. Let $X_1, X_2, \cdots, X_{n_1}$ be a sample from $N(\xi, \sigma^2)$ and $Y_1, Y_2, \cdots, Y_{n_2}$ a sample from $N(\eta, \tau^2)$. We wish to test the hypothesis $H: \xi = \eta$ against the alternatives $K: \xi \neq \eta$ where σ^2 and τ^2 are unknown nuisance parameters. The test usually proposed is of the form: Accept H whenever

$$a < (\bar{Y} - \bar{X})/[s_{\bar{x}}^2 + s_{\bar{y}}^2]^{\frac{1}{2}} < b \tag{6}$$

where $s_{\bar{x}}^2 = s_x^2/n_1$, $s_{\bar{y}}^2 = s_y^2/n_2$ and (a, b) is some interval containing zero.

If one divides the numerator and denominator by $(\sigma^2/n_1 + \tau^2/n_2)^{\frac{1}{2}}$, then the statistic in (6) has the form $X/h(Z)$ where X is distributed $N(\delta, 1)$ with $\delta = (\eta - \xi)/(\sigma^2/n_1 + \tau^2/n_2)^{\frac{1}{2}}$, $h(z) = z^{\frac{1}{2}}$, and Z is distributed as

$$[\lambda_1/(n_1 - 1)]\chi_1^2(n_1 - 1) + [\lambda_2/(n_2 - 1)]\chi_2^2(n_2 - 1) \tag{7}$$

with $\lambda_1 = (\sigma^2/n_1)/(\sigma^2/n_1 + \tau^2/n_2)$, and $\lambda_2 = 1 - \lambda_1$. If we take $m_i = n_i - 1$, and $\alpha_1 = \alpha_2 = \cdots = \alpha_{m_1} = \lambda_1/m_1$, $\alpha_{m_1+1} = \cdots = \alpha_{m_1+m_2} = \lambda_2/m_2$, and recall that a $\chi^2(n)$ variable can be decomposed into the sum of n independent $\chi^2(1)$ variables, then (7) can be rewritten as

$$\textstyle\sum_{k=1}^m \alpha_k \chi_k^2(1), \qquad \alpha_k \geqq 0, \quad \sum_{k=1}^m \alpha_k = 1, \tag{8}$$

where $m = m_1 + m_2$.

But σ^2 and τ^2 are unknown; thus, the probability of the event described in (6) cannot be determined. None-the-less if the theorem of Section 4 applies, then we can at least obtain bounds for this probability. In fact, we shall prove

Corollary 1. *Let δ be as above, then for $|\delta| \leqq 2$ and for any interval (a, b) containing zero*

$$\begin{aligned} P\{a < t_\nu(\delta) < b\} &\leqq P\{a < (\bar{Y} - \bar{X})/[s_x^2 + s_y^2]^{\frac{1}{2}} < b\} \\ &\leqq P\{a < t_m(\delta) < b\}. \end{aligned} \tag{9}$$

where $t_n(\delta)$ denotes a random variable having the non-central t-distribution with n degrees of freedom and non-centrality parameter δ. Here $m = n_1 + n_2 - 2$ and ν is any whole number not exceeding $\min[m_1/\lambda_1, m_2/\lambda_2]$; *for example,* $\nu = \min[n_1 - 1, n_2 - 1]$.

Proof. Clearly the denominator Z has the structure (2) by reason of (8). If we can show that $X/Z^{\frac{1}{2}}$ satisfies Condition (A) for $|\delta| \leqq 2$, then the above mentioned corollary follows immediately from the theorem of Section 4 once we note that, in the present case, $\bar{V}_{(r)}$ is distributed as $\chi^2(r)/r$.

Let us first assume that $\delta \geqq 0$, we then have for any $a < 0$,

$$g_a(z) = P\{a < X/z^{\frac{1}{2}} < 0\} = (2\pi)^{-\frac{1}{2}}\textstyle\int_{az^{1/2}}^0 \exp[-(r - \delta)^2/2]\, dr,$$

so that

$$dg_a(z)/dz = -a \exp[-(-az^{\frac{1}{2}} + \delta)^2/2]/2(2\pi z)^{\frac{1}{2}} > 0.$$

Now the derivative of $g_a(z)$ is clearly a decreasing, positive function on $[0, \infty)$ since the denominator is increasing on this region, and the numerator is decreasing. Hence, $g_a(z)$ is concave on $[0, \infty)$.

Next we consider, for $b > 0$,

$$g_b(z) = P\{0 < X/z^{\frac{1}{2}} < b\} = (2\pi)^{-\frac{1}{2}} \int_0^{bz^{\frac{1}{2}}} \exp[-(r - \delta)^2/2]\, dr.$$

In this case

$$dg_b(z)/dz = b \exp[-(bz^{\frac{1}{2}} - \delta)^2/2]/2(2\pi z)^{\frac{1}{2}} > 0,$$

and

$$(10) \qquad d^2g_b(z)/dz^2 = \{-\exp[-(bz^{\frac{1}{2}} - \delta)^2/2]/4z^{\frac{3}{2}}\}(b^2z - b\delta z^{\frac{1}{2}} + 1).$$

The first factor on the right side of (10) is always negative when $z \geqq 0$, and the sign of the second derivative depends only on the sign of the polynomial term. $g_b(z)$ is concave when the second derivative is less than or equal to zero; that is, when

$$(11) \qquad b^2z - b\delta z^{\frac{1}{2}} + 1 \geqq 0 \quad \text{for all} \quad z \geqq 0.$$

Now (11) holds whenever $(b^2\delta^2 - 4b^2) \leqq 0$, or equivalently $\delta \leqq 2$. A similar proof goes through for $\delta < 0$. Thus, we see that Condition (A) is satisfied whenever $|\delta| \leqq 2$.

As an illustration of the type of bounds that may be obtained, consider the case where $n_1 = n_2 = 10$. We are interested in testing $\xi = \eta$ against the one-sided alternatives $\xi < \eta$. One rejects whenever

$$(10)^{\frac{1}{2}}(\bar{Y} - \bar{X})/[s_X^2 + s_Y^2]^{\frac{1}{2}} > c.$$

Let $\beta(\delta)$ denote the power curve for this test with $\delta = (10)^{\frac{1}{2}}(\eta - \xi)/(\sigma^2 + \tau^2)^{\frac{1}{2}}$. If one takes $c = 1.4$ and notes that $\beta(0)$ is type I error, then

$$0.0893 \leqq \beta(0) \leqq 0.0975,$$

$$0.7170 \leqq \beta(2) \leqq 0.7171.$$

Hence, we have a test which guarantees that the type I error will not exceed 0.0975 and, at the same time, its power against alternatives $\delta \geqq 2$ is at least 0.7170.

6. Application to Lord's u-test. Lord [2] proposed a modified t-test, called the u-test, in which the sample range W replaces s as the estimate of standard deviation. He suggests that, when the sample sizes are small, the simplicity and ease of calculation of the u-statistic more than compensate for the slight loss of efficiency.

Let $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ be two samples of size n from $N(\xi, \sigma^2)$ and $N(\eta, \tau^2)$ respectively. We shall now accept the hypothesis H:

$\xi = \eta$ whenever

$$|\bar{Y} - \bar{X}|/(W_1^2 + W_2^2)^{\frac{1}{2}} < c, \tag{12}$$

where W_1 and W_2 are the ranges of the X and Y samples. Under the null hypothesis, the statistic in (12) and the statistic in (6) differ only in the form of the variance estimate Z. In the present case, division of the numerator and denominator by $(\sigma^2 + \tau^2)^{\frac{1}{2}}$ causes the statistic in (12) to have the form $X/h(Z)$ where X is distributed $\mathrm{N}(0, 1/n)$, and Z is distributed as

$$\lambda W_1^{*2}(n) + (1 - \lambda) W_2^{*2}(n) \tag{13}$$

where $\lambda = \sigma^2/(\sigma^2 + \tau^2)$ and $W^*(n)$ denotes a random variable having the distribution of the range of a sample of size n from a standard normal population. Note that $W_1^*(n)$ and $W_2^*(n)$ are identically distributed only when we have equal sample sizes. We can apply the theorem of Section 4 to obtain

COROLLARY 2. *Under the null hypothesis* $\xi = \eta$ *we have*

$$\begin{aligned} P\{|V/n^{\frac{1}{2}}|/W^*(n) < c\} &\leqq P\{|\bar{Y} - \bar{X}|/(W_1^2 + W_2^2)^{\frac{1}{2}} < c\} \\ &\leqq P\{|(2/n)^{\frac{1}{2}}V|/(W_1^{*2}(n) + W_2^{*2}(n))^{\frac{1}{2}} < c\}, \end{aligned} \tag{14}$$

where V has the standard normal distribution and $W^*(n)$ is defined as in (13).

The percentage points of the distribution involved in the lower bound are given in Table 9 of [2]. Unfortunately the distribution involved in the upper bound is untabulated. However, we may note that, since the range is a non-negative random variable, $[W_1^*(n) + W_2^*(n)]^2$ is never less than $W_1^{*2}(n) + W_2^{*2}(n)$. So that we have

$$\begin{aligned} &P\{|(2/n)^{\frac{1}{2}}V|/(W_1^{*2}(n) + W_2^{*2}(n))^{\frac{1}{2}} < c\} \\ &\qquad \leqq P\{|(2/n)^{\frac{1}{2}}V|/W_1^*(n) + W_2^*(n) < c\}. \end{aligned} \tag{15}$$

Lord presents the percentage points of $2|(2/n)^{\frac{1}{2}}V|/(W_1^*(n) + W_2^*(n))$ in Table 10. So we can once again obtain bounds for the type I error. For example, with $n = 10$, $c = 0.288$ we have $0.001 \leqq \alpha \leqq 0.02$.

It would be possible to extend the bounds to the power curve, as was done in Section 5. It is possible also to extend the bounds obtained in this section to the case where one has unequal sample sizes. This extension requires more machinery and will be discussed in a future paper.

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