ON AN OPERATOR LIMIT THEOREM OF ROTA

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1. Introduction. Let (X, Σ, μ) be a probability measure space and let $f \varepsilon L^p(X, \Sigma, \mu)$ for some $p > 1$. Given a sequence $P_1, P_2, \cdots$ of doubly stochastic operators on $L^1(X, \Sigma, \mu)$, Rota [4] has shown that $\lim_{n\to\infty} P_1^*P_2^* \cdots P_n^*P_nP_{n-1} \cdots P_1f$ exists a.e. (Convergence in the L^p norm also holds, by a variety of proofs ($1 \leqq p < \infty$). Almost everywhere convergence does not extend to the case $p = 1$ [1].) In the same article it was stated that a convergence theorem using the reverse index product $P_n^*P_{n-1}^* \cdots P_1^*P_1P_2 \cdots P_n$ was a possible generalization. The impossibility of such a result, or even of a strongly continuous inhomogeneous semi-group analog thereof, is shown in this note. We thank D. L. Burkholder for permission to include, as a second example, a result he discovered in connection with a distinct problem.

By a *doubly stochastic operator* on $L^1(X, \Sigma, \mu)$ is meant a linear operator on $L^1(X, \Sigma, \mu)$ into itself such that

(1) $\int |Pf| \, d\mu \leqq \int |f| \, d\mu$,

(2) $f \geqq 0$ a.e. $\Rightarrow Pf \geqq 0$ a.e., and

(3) $P1 = 1$ a.e., where 1 is the constant function assuming everywhere the value 1.

It is readily shown that the adjoint operator P^* satisfies (2) and (3) and does not increase L^1 norms of L^∞ functions. Thus P^* may be extended to an operator on L^1 and this extension, denoted also P^*, is doubly stochastic.

We say a family $\{P(t, s)\}_{\{t \geqq s \geqq 0\}}$ of bounded operators on a Banach space B is an *inhomogeneous semi-group* if $P(t, r)P(r, s) = P(t, s)$ for $t \geqq r \geqq s$ and $P(t, t) = I$, the identity operator. The semi-group is *strongly continuous* if for each $b \varepsilon B$, $P(t, s)b$ is a continuous function on the subset $\{t \geqq s \geqq 0\}$ of R^2 to B.

Given $f \varepsilon L^1(X, \Sigma, \mu)$, $E\{\cdot \mid f\}$ denotes conditional expectation with respect to the σ-field determined by f.

2. Continuous parameter example. Let $\{P(t, s)\}$ be an inhomogeneous semi-group of doubly stochastic operators on $L^1(X, \Sigma, \mu)$. Using the separability theory of Doob [2], the following version of Rota's theorem can be proved: Given $p > 1$ and $f \varepsilon L^p(X, \Sigma, \mu)$, the family $\{P^*(t, 0)P(t, 0)f\}_{t \varepsilon [0, \infty)}$ can be redefined for each t on a set of μ-measure zero in such a manner that $\lim_{t\to\infty} P^*(t, 0)P(t, 0)f$ exists everywhere. If one reverses the operations and considers the limiting behavior of $P(t, 0)P^*(t, 0)f$ as $t \to \infty$, pointwise convergence need not hold, even if $\{P(t, s)\}$ is *strongly continuous* in each L^p ($1 \leqq p < \infty$). This we now show.

Let (X, Σ, μ) be the unit circle with normalized Lebesgue measure: $d\mu =$

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$d\theta/2\pi$. In the expressions $\theta I_{(a,b)}$, θ denotes the argument function on the circle and $I_{(a,b)}$ the indicator function of the arc (a, b). Letting $n = 0, 1, \cdots$ and $f \varepsilon L^1(X, \Sigma, \mu)$, define

$$P(t, s)f(\theta) = f(\theta + \pi(t - s)) \qquad \text{for } 2n - 1 \leqq s \leqq t \leqq 2n;$$

$$P(t, s)f = E\{f \mid \theta I_{(-s\pi, (2-t)\pi)}\} \qquad \text{for } 4n \leqq s \leqq t \leqq 4n + 1;$$

and

$$P(t, s)f = E\{f \mid \theta I_{(t\pi, (s+2)\pi)}\} \qquad \text{for } 4n + 2 \leqq s \leqq t \leqq 4n + 3.$$

Denoting $P(n, n - 1)$ by P_n, it is evident that

P_{2n} is rotation through π radians,

$P_{4n+1} = E\{\cdot \mid \theta I_{(0,\pi)}\}$,

$P_{4n+3} = E\{\cdot \mid \theta I_{(-\pi,0)}\}$, and

$P(n, 0)P^*(n, 0) = P_n P_{n-1} \cdots P_1 P_1^* P_2^* \cdots P_n^*$.

Hence

$$P(4n + 1, 0)P^*(4n + 1, 0) = E\{\cdot \mid \theta I_{(0,\pi)}\}$$

and

$$P(4n + 3, 0)P^*(4n + 3, 0) = E\{\cdot \mid \theta I_{(-\pi,0)}\}.$$

Thus as $t \to \infty$ neither pointwise a.e. nor L^p norm convergence need hold for $\{P(t, 0)P^*(t, 0)f\}_{t \varepsilon [0,\infty)}$ (we assume the process is separable). It is easy to see that $\{P(t, s)\}$ is strongly continuous in each L^p space $(1 \leqq p < \infty)$.

3. Discrete example using only conditional expectations. In the following example, due to Burkholder, all the P_k are conditional expectation operators. Besides being doubly stochastic, conditional expectation operators are self-adjoint, idempotent, and are Hilbert space positive-definite.

Let (X, Σ, μ) be a probability space on which there are defined independent random variables f, g each having the normal distribution with mean zero and variance one. Define

$$h_\theta = (\cos\theta)f + (\sin\theta)g.$$

Since linear combinations of normal random variables are normal, the h_θ are normal. Since h_θ and $h_{\theta+\pi/2}$ are orthogonal, they are independent. Finally, since $h_\theta = \cos(\theta - \varphi)h_\varphi + \cos(\pi/2 - (\theta - \varphi))h_{\varphi+\pi/2}$, it follows that

$$E\{h_\theta \mid h_\varphi\} = \cos(\theta - \varphi)h_\varphi .$$

Setting $P_k = E\{\cdot \mid h_{\theta_k}\}$, one obtains

$$P_n^* P_{n-1}^* \cdots P_1^* P_1 P_2 \cdots P_n f = \cos\theta_n \cos^2(\theta_n - \theta_{n-1}) \cdots \cos^2(\theta_2 - \theta_1)h_{\theta_n}.$$

Now since $|\cos(\pi/2n)| > 1 - 2/n^2$, and $(1 - 2/n^2)^{2n-2} \to 1$ as $n \to \infty$, there exist sequences $\{\theta_n\}_{n=1}^{\infty}$ which contain both 0 and $\pi/2$ infinitely often and satisfy $\prod_{n=2}^{\infty} \cos^2(\theta_n - \theta_{n-1}) > \frac{1}{2}$. For such a sequence $|P_n{}^*P_{n-1}^* \cdots P_1{}^*P_1P_2 \cdots P_nf|$ is equal to 0 for infinitely many n and greater than $|f|/2$ for infinitely many n. Thus neither a.e. nor L^p norm convergence holds.

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