PROPERTIES OF THE MEDIAN AND OTHER ORDER STATISTICS OF LOGISTIC VARIATES

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1. Introduction and summary. Recently, increasing use has been made of the logistic distribution. In addition to the many investigations of the logistic as a regression curve, several authors have investigated the properties and suggested the use of the logistic as a density function. One reason for this is the property that the inverse of the logistic distribution function can be expressed as the logarithm of a rational function. A general background for the work in this field can be gained by reading references [1], [2], [3], [7], and [10]. In this paper several additional properties of a sample of logistic variates will be described.

In Section 2, it will be shown that the logistic distribution and its moment generating function can be expressed as a Maclaurin series where the coefficients are simple functions of Bernoulli numbers. In Section 3, the moment generating function of the median is given. In Section 4, the variance of the median will be determined and the relative efficiency of the median of logistic variates to the mean for various sample sizes as well as the asymptotic efficiency is given. In Section 5, the cumulant generating function of the median of logistic variates is derived and the cumulants, themselves, given in terms of zeta values. In Section 6, the variance and covariance of any two order statistics are given.

The order statistic variance-covarance matrix can be used to determine the BLUE estimators of the population location and scale parameters. These may be expected to have high efficiency for the estimation of logistic parameters [4]. Although the maximum likelihood estimate of the population location parameter $\hat{\theta}$ satisfies the relationship,

$$\tfrac{1}{2} = [\textstyle\sum_{i=1}^{n} F(x_i - \hat{\theta})]/n,$$

iterative methods must be used to solve for $\hat{\theta}$. Therefore the computational advantage of the BLUE estimates as well as their probable efficiency would tend to make them useful in the case of the logistic distribution.

2. Expression for the logistic distribution and its moment generating function in terms of Bernoulli numbers. The cumulative distribution function of the logistic distribution in its reduced form is defined by the expression, $F(x) = 1/(1 + e^{-x})$ for $-\infty < x < \infty$ and the density function is $f(x) = e^{-x}/(1 + e^{-x})^2$. The distribution is symmetric about 0 and graphically resembles the normal distribution. It possesses the following properties listed by Gumbel [7],

$$f(x) = F(x)[1 - F(x)] \tag{1}$$

$$x = \ln [F(x)/(1 - F(x))] \tag{2}$$

$$\sigma^2 = \pi^2/3 \tag{3}$$

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The cumulative distribution function of the logistic distribution can be expressed as

$$F(x) = e^x/(1 + e^x) \tag{4}$$

or

$$F(x) = \tfrac{1}{2} + \textstyle\sum_{j=1}^{\infty} [(2^{2j} - 1)/(2j)!]B_{2j}x^{2j-1} \tag{5}$$

where B_j are Bernoulli numbers as defined by Knopp [10] ($B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}, \cdots$).

The moment generating function, MGF (t), of the logistic distribution is given by Talacko [14] as

$$\text{MGF}(t) = \pi t/\sin \pi t \qquad \text{for} \quad |t| < 1 \tag{6}$$

which can be expressed in terms of Bernoulli numbers as

$$\text{MGF}(t) = \textstyle\sum_{j=0}^{\infty} (-1)^{j-1}[2(2^{2j-1} - 1)/(2j)!]B_{2j}(\pi t)^{2j} \tag{7}$$

or

$$\text{MGF}(t) = \textstyle\sum_{j=0}^{\infty} [(-1)^j/(2j)!]B_{2j}(\tfrac{1}{2})(2\pi t)^{2t} \tag{8}$$

for the Bernoulli polynomial in terms of $B_n(\frac{1}{2})$. Therefore the $2j$th moment is $|B_{2j}(\frac{1}{2})|(2\pi)^{2j}$.

3. Moment generating function of the sampling distribution of the median of logistic variates. The density function of the median, x_m, of a random sample of size n (n odd) for any density function, $f(x)$, equals

$$g(x_m) = [n!/(k!)^2]f(x)[F(x)(1 - F(x))]^k \tag{9}$$

where $k = \frac{1}{2}(n - 1)$. Considering a logistic distribution with a scale parameter, b,

$$F(x) = 1/(1 + e^{-x/b}), \tag{10}$$

the moment generating function of the sampling distribution of the median can be expressed as

$$\text{MGF}(t) = [\textstyle\prod_{i=1}^{\infty} (1 - (b^2t^2/(k + i)^2))]^{-1}. \tag{11}$$

Similarly, the characteristic function can be written as

$$\text{CF}(t) = [\textstyle\prod_{i=1}^{\infty} (1 + (b^2t^2/(k + i)^2))]^{-1} \tag{12}$$

using an infinite product expansion of the hyperbolic secant.

4. Variance of the median and its relative and asymptotic efficiency. The variance of the median can be found directly by taking the second derivative of the moment generating function of the median

$$d^2\,\text{MGF}(t)/dt^2\,|_{t=0} = 2b^2[\tfrac{1}{6}\pi^2 - \textstyle\sum_{i=1}^{k} i^{-2}] \tag{13}$$

which equals the variance of the median of the logistic distribution. This can be approximated by replacing the summation operation by integration and therefore

(14) $$\text{Var (median)} \doteq 4b^2/n.$$

The variance of the mean of logistic variates is

$$\text{Var (mean)} = \pi^2 b^2/3(2k + 1).$$

Using (13), the inverse of the efficiency is

(15) $$E^{-1} = 6(2k + 1)/\pi^2[\tfrac{1}{6}\pi^2 - \textstyle\sum_{i=1}^{k} i^{-2}].$$

It can be shown that $E^{-1} \to 12/\pi^2$ as $k \to \infty$ and consequently $E \to \pi^2/12$ or .8225. This is a greater relative efficiency than a similar comparison using the mean and median of the normal distribution where $E = .64$.

Table 1 lists the relative efficiency of the median of the logistic variates, as obtained from (15) for odd n between 3 and 19 and $n = 51$ and 101. It can be seen from Table 1 that E rapidly approaches its asymptotic efficiency. The asymptotic efficiency of the median relative to the Cramér-Rao lower bound is

(16) $$\text{E (median)} = (\pi^2/12)(9/\pi^2) = .75.$$

TABLE 1

Relative efficiency of the median when compared with the mean for the logistic distribution and normal distribution

n	For Logistic Dist.	For Normal Dist.
3	.85018	.743
5	.83302	.697
7	.82795	.679
9	.82581	.669
11	.82471	.663
13	.82408	.659
15	.82368	.656
17	.82341	.653
19	.82322	.651
51	.82257	
101	.82249	
∞	.82247	.637

5. Cumulants. In this section, the cumulants of the median of logistic variates will be given. Taking logarithms of (11)

(17) $$\text{CGF median } (t) = -\textstyle\sum_{i=1}^{\infty} \ln [1 - (b^2t^2/(k + 1)^2)].$$

Using the Maclaurin expansion of $\ln (1 - x)$ and reversing the order of summation,

(18) $$\text{CGF median } (t) = \textstyle\sum_{j=1}^{\infty} [(b^2t^2)^j/j]\zeta(2j, k + 1)$$

where $\zeta(2j, k+1)$ is a generalized zeta function [15]. Thus, the $2j$th cumulant is

$$(b^{2j}/j)(2j)! = \zeta(2j, k+1). \tag{19}$$

Further, it can be shown that CGF $[(n-1)^{\frac{1}{2}}t] \to 2b^2t^2$ as $n \to \infty$ and, therefore, all cumulants except the second approach zero. Thus agreeing with theory, Cramér [6], the distribution of $(n-1)^{\frac{1}{2}}$ times the median approaches the normal distribution.

6. Covariances of logistic order statistics. The covariance of any two order statistics of a random sample of logistic variates can be derived by using a method similar to that used in Section 4.

The covariance of the Kth and Lth order statistics for $L > K$ is

$$\operatorname{Cov}(x_K, x_L) = E(x_K x_L) - E(x_K)E(x_L). \tag{20}$$

To evaluate $E(x_K x_L)$, a variation of the usual moment generating function of the joint distribution of the K and L order statistics will be used. From Sarhan and Greenberg [12] the joint distribution for $L > K$ is

$$\begin{aligned} F(x_K, x_L) &= [n!\, F(x_K)^{K-1}/(K-1)!\,(L-K-1)!\,(n-L)!] \\ &\quad \cdot [F(x_L) - F(x_K)]^{L-K-1}[1 - F(x_L)]^{n-L} f(x_K) f(x_L)\, dx_K\, dx_L \quad \text{for} \quad x_K < x_L. \end{aligned} \tag{21}$$

Using (21), the moment generating function of the joint density of order statistics x_K and x_L can be written as

$$\begin{aligned} \operatorname{MGF}(t_K, t_L) &= C \int_{-\infty}^{\infty} \int_{-\infty}^{x_L} \exp(t_K x_K + t_L x_L) F(x_K)^{K-1} \\ &\quad \cdot [F(x_L) - F(x_K)]^{L-K-1}[1 - F(x_L)]^{n-L} f(x_K) f(x_L)\, dx_K\, dx_L \end{aligned} \tag{22}$$

where $C = n!/(K-1)!\,(L-K-1)!\,(n-L)!$.

The expression for MGF (t_K, t_L) given in (22) is difficult to evaluate directly. However, the partial derivative

$$\begin{aligned} \partial \operatorname{MGF}(t_K, t_L)/\partial t_K &= C \int_{-\infty}^{\infty} \int_{-\infty}^{x_L} x_K e^{t_L x_L} [F(x_K)]^{K-1} \\ &\quad \cdot [F(x_L) - F(x_K)]^{L-K-1}[1 - F(x_L)]^{n-L} f(x_K) f(x_L)\, dx_K\, dx_L \end{aligned} \tag{23}$$

can be evaluated. By using the transformations $y_K = F(x_K)$ and $y_L = F(x_L)$, (23) can be expressed as

$$\begin{aligned} \partial \operatorname{MGF}(t_K, t_L)/\partial t_K &= C \int_0^1 (1 - y_L)^{n-L} [y_L/(1 - y_L)]^{t_L} \\ &\quad \cdot \int_0^{y_L} F^{-1}(y_K) y_K^{K-1} (y_L - y_K)^{L-K-1}\, dy_K\, dy_L. \end{aligned} \tag{24}$$

Since $F^{-1}(y_K) = \log [y_K/(1 - y_K)] = -2 \sum_{i=0}^{\infty} (1 - 2y_K)^{2i+1}/(2i+1)$, which converges uniformly for all $0 < y_K < 1$, (24) may be expressed as

$$\begin{aligned} \partial \operatorname{MGF}(t_K, t_L)/\partial t_K &= -2C \sum_{i=0}^{\infty} 1/(2i+1) \int_0^1 (1 - y_L)^{n-L} \\ &\quad \cdot [y_L/(1 - y_L)]^{t_L} \int_0^{y_L} (1 - 2y_K)^{2i+1} y_K^{K-1} (y_L - y_K)^{L-K-1}\, dy_K\, dy_L. \end{aligned} \tag{25}$$

The expression $(1 - 2y_K)^{2i+1}$ may be represented as

$$\sum_{j=0}^{2i+1} (-1)^j 2^j \binom{2i+1}{j} y_K^{\,j}. \tag{26}$$

Let G represent $\int_0^{y_L} (1 - 2y_K)^{2i+1} y_K^{K-1}(y_L - y_K)^{L-K-1}\, dy_K$. Since a finite sum may be integrated term by term,

$$G = \sum_{j=0}^{2i+1} (-1)^j 2^j \binom{2i+1}{j} \int_0^{y_L} y_K^{K+j-1}(y_L - y_K)^{L-K-1}\, dy_K . \tag{27}$$

The integral in (27), $\int_0^{y_L} y_K^{K+j-1}(y_L - y_K)^{L-K-1}\, dy_K$, equals $y_L^{L+j-1}\beta(K + j, L - K)$, where β represents the beta function. Therefore,

$$\begin{aligned} \partial\, \text{MGF}\,(t_K, t_L)/\partial t_K &= -2C \sum_{i=0}^{\infty} (2i + 1)^{-1} \sum_{j=0}^{2i+1} (-1)^j 2^j \binom{2i+1}{j} \\ &\quad \beta(K + j, L - K) \int_0^1 y_L^{L+j+t_L-1}(1 - y_L)^{n-L-t_L}\, dy_L \end{aligned} \tag{28}$$

or

$$\begin{aligned} &= -2C \sum_{i=0}^{\infty} (2i + 1)^{-1} \sum_{j=0}^{2i+1} (-1)^j 2^j \binom{2i+1}{j} \beta(K + j, L - K) \\ &\quad \beta(L + j + t_L, n - L - t_L + 1). \end{aligned} \tag{29}$$

Assuming that the series may be differentiated term by term, the product moment $E(x_K x_L)$ equals

$$\begin{aligned} -2C \sum_{i=0}^{\infty} (2i + 1)^{-1} \sum_{j=0}^{2i+1} (-1)^j 2^j \binom{2i+1}{j} \beta(K + j, L - K) \\ (d/dt_L)\beta(L + j + t_L, n - L + 1 - t_L)\big|_{t_L=0}. \end{aligned} \tag{30}$$

The derivative $(d/dt_L)\beta(L + j + t_L, n - L + 1 - t_L)$ equals

$$1/(n + j)!\, (d/dt_L)\Big[\prod_{i=1}^{L+j-1} (i + t_L) \prod_{i=1}^{n-L} (i - t_L)(t_L\pi/\sin \pi t_L)\Big]. \tag{31}$$

However, $t_L\pi/\sin \pi t_L$ is the moment generating function of the logistic distribution (6) and therefore its derivative evaluated at zero is zero and its limit as $t_L \to 0$ is 1. Therefore the derivative evaluated at zero is

$$\Big[\sum_{i=1}^{L+j-1} i^{-1} - \sum_{i=1}^{n-L} i^{-1}\Big](L + j - 1)!\, (n - L)!/(n + j)!.$$

The expression for $E(x_K x_L)$, when simplified is

$$\begin{aligned} E(x_K x_L) = -2 \sum_{i=0}^{\infty} (2i + 1)^{-1} \sum_{j=0}^{2i+1} (-2)^j \binom{2i+1}{j} \\ \Big[\sum_{i=1}^{L+j-1} i^{-1} - \sum_{i=1}^{n-L} i^{-1}\Big]\binom{K+j-1}{K-1}\Big/\binom{n+j}{n}. \end{aligned} \tag{32}$$

As recently as 1963 it was stated that "It would be desirable to augment 'asymptotic calculations' with exact computations of covariances of some logistic order statistics, Cov (x_i, x_j), evidently numerical integration is required, and this has not been undertaken," [3]. Actually, by means of recurrence formulas, Kjelsberg in 1962 obtained exact numerical results for the covariances of logistic order statistics from samples of size five or less [9]. In the next three sections, (32) is reduced to a finite sum which can be easily used to evaluate logistic product moments for any sample size.

In Section 7 a lemma is derived which can be used to evaluate series of the type $\sum_{i=0}^{\infty} \Delta^i F(0)/i$ where Δ^i is the ith forward difference from $F(0)$. In Section 8 the expression $\sum_{i=0}^{\infty}(-1)^i\Delta^i F(0)/i$ is evaluated where the terms $F(2i+1)$ are the coefficients of $(2i+1)^{-1}$ in (32). Finally, in Section 9 the series, simplified by combining the results of Sections 7 and 8, will be expressed as a finite sum.

7. Reverse use of Euler's transformation. Euler's transformation may be expressed as

$$\sum_{s=0}^{\infty}[F(s)/t^{s+1}] = \sum_{s=0}^{\infty}[\Delta^s F(0)/(t-1)^{s+1}]. \tag{33}$$

By multiplying the left side of (33) by t and the right side by $(t-1)+1$ and simplifying one obtains

$$\sum_{s=1}^{\infty}[\Delta^s F(0)/(t-1)^s] = \sum_{s=1}^{\infty}F(s)[1/t^s - 1/t^{s+1}] - [F(0)/t]. \tag{34}$$

By substituting $t \equiv ve^{u+1}$ and integrating with respect to u from zero to infinity, one obtains

$$\sum_{s=1}^{\infty}[\Delta^s F(0)/sv^s] = \sum_{s=1}^{\infty}[F(s)/s(v+1)^s] - F(0)\log[(v+1)/v] \tag{35}$$

or with $v = 1$

$$\sum_{s=1}^{\infty}[\Delta^s F(0)/s] = \sum_{s=1}^{\infty}[F(s)/s2^s] - F(0)\log 2. \tag{36}$$

8. Evaluating the alternating series in the special case. $E(x_K x_L)$ was obtained by differentiating the expression

$$C\int_0^1 (1-y_L)^{n-L}(y_L/(1-y_L))^{t_L}\int_0^{y_L}F^{-1}(y_K){y_K}^{K-1}(y_L-y_K)^{L-K-1}\,dy_K\,dy_L \tag{37}$$

with respect to t_L and evaluating the derivative at zero. The relationship

$$\log[y_K/(1-y_K)] = -2\sum_{i=0}^{\infty}(1-2y_K)^{2i+1}/(2i+1) \tag{38}$$

forms the series of *odd terms* of (32). The series

$$-2\sum_{i=1}^{\infty}(1-2y_K)^i/i = 2\log 2y_K = F^{*-1}(y_K). \tag{39}$$

If $\exp[t_K F^{*-1}(y_K)] = \exp[2t_K\log 2y_K]$ is substituted for $F^{-1}(y_K)$ in (37) the expression may be evaluated as

$$2^{t_K-1}C\beta(K+t_K, L-K)\beta(n-L-t_L+1, L+t_L+t_K) \tag{40}$$

which when differentiated with respect to t_L and t_K yields

$$2((\pi^2/6) - \sum_{i=1}^{L-1}i^{-2} + [\log 2 + \sum_{i=1}^{K-1}i^{-1} - \sum_{i=1}^{n}i^{-1}][\sum_{i=1}^{L-1}i^{-1} - \sum_{i=1}^{n-L}i^{-1}]). \tag{41}$$

9. The expression for $E(x_K x_L)$ as a finite sum. By combining (36) and (41), $E(x_K x_L)$ can be expressed as

$$(\pi^2/6) - \sum_{i=1}^{L-1}i^{-2} + [\sum_{i=1}^{K-1}i^{-1} - \sum_{i=1}^{n}i^{-1}][\sum_{i=1}^{L-1}i^{-1} - \sum_{i=1}^{n-L}i^{-1}] + \sum_{i=1}^{\infty}[(K)_i/i(n+1)_i][\sum_{i=1}^{L+i-1}i^{-1} - \sum_{i=1}^{n-L}i^{-1}] \tag{42}$$

where $(K)_i$ is Pochammer's symbol for $K(K+1)\cdots(K+j-1)$. Since

$$(K)_i/(n+1)_i = (n-K+1)\binom{n}{K-1}\sum_{s=0}^{n-K}\binom{n-K}{s}[(-1)^s/K+i+s], \tag{43}$$

the infinite series in (42) may be written as

$$(44)\quad (n - K + 1)\binom{n}{K-1} \sum_{s=0}^{n-K} \binom{n-K}{s}(-1)^s \sum_{i=1}^{\infty} \{[\sum_{j=1}^{L+i-1} j^{-1} - \sum_{j=1}^{n-L} j^{-1}]/i(K + i + s)\}.$$

The expression

$$(45)\quad \sum_{i=1}^{\infty} [(K + s + i)i]^{-1} \sum_{j=1}^{L+i+1} j^{-1}$$

$$(46)\quad = (K + s)^{-1} \sum_{i=1}^{\infty} [i^{-1} - (K + i + s)^{-1}] \sum_{j=1}^{L+i-1} j^{-1}$$

$$(47)\quad = (K + s)^{-1}[\sum_{i=1}^{K+s} i^{-1} \sum_{j=1}^{L+i-1} j^{-1} + \sum_{i=1}^{\infty} (K + s + i)^{-1} \sum_{j=L+i}^{K+L+s+i-1} j^{-1}]$$

$$(48)\quad = (K + s)^{-1} \cdot [\sum_{i=1}^{K+s} i^{-1} \sum_{j=1}^{L+i-1} j^{-1} + \sum_{j=0}^{K+s-1} \sum_{i=1}^{\infty} [(K + s + i)(L + j + i)]^{-1}].$$

Since

$$(49)\quad \sum_{i=1}^{\infty} [(K + s + i)(L + j + i)]^{-1} = [(K + s) - (L + j)]^{-1}[\sum_{i=1}^{K+s} i^{-1} - \sum_{i=1}^{L+j} i^{-1}],$$

if $K + s \neq L + j$ and otherwise $= (\pi^2/6) - \sum_{i=1}^{K+s} i^{-2}$. Series (42) has been reduced to a finite sum and

$$\begin{aligned} &E(x_K x_L) \\ (50)\quad &= (\pi^2/6) - \sum_{i=1}^{L-1} i^{-2} + [\sum_{i=1}^{K-1} i^{-1} - \sum_{i=1}^{n} i^{-1}][\sum_{i=1}^{L-1} i^{-1} - \sum_{i=1}^{n-L} i^{-1}] \\ &\quad + (n - K + 1)\binom{n}{K-1} \sum_{s=0}^{n-K} \binom{n-K}{s}(-1)^s(K + s)^{-1} \\ &\quad \cdot [\sum_{i=1}^{K+s} i^{-1} \sum_{j=1}^{L+i-1} j^{-1} + \sum_{j=0}^{K+s-1} \sum_{i=1}^{\infty} [(K + s + i)(L + j + i)]^{-1} \\ &\quad \cdot \sum_{j=1}^{n-L} j^{-1} \sum_{j=1}^{K+s} j^{-1}]. \end{aligned}$$

Exact numerical values for $E(x_K x_L)$ have been obtained by means of this formula. These values were found to be exactly equal to the values for n less than five obtained by Kjelsberg using recurrence relations and Monte Carlo generated values obtained by the authors for larger sample sizes.

The cumulant generating function of the Kth order statistic, $\mathrm{CGF}_K(t)$, for $K >$ median can be expressed in terms of cumulant generating function of the median.

$$(51)\quad \mathrm{CGF}_K(t) = \mathrm{CGF}_{\text{median}}(t) + \sum_{i=(n+1)/2}^{K-1} \log[1 + (bt/i)] - \sum_{i=n-K+1}^{(n-1)/2} \log[1 - (bt/i)].$$

The first derivative of (51) at $t = 0$ is

$$(52)\quad E(x_K) = \partial\, \mathrm{CGF}_{\text{median}}(t)/\partial t\,|_{t=0} + \sum_{i=n-K+1}^{K-1} (b/i)$$

or

$$(53)\quad E(x_K) = \sum_{i=n-K+1}^{K-1} (b/i).$$

By symmetry,

$$(54)\quad E(x_{n-K+1}) = -E(x_K).$$

Thus, the covariance of the K and L order statistic can be found by substituting (50) and (53) and/or (54) into (20).

Also using the second derivative of the cumulant generating function, the variance of the Kth order statistic can be found.

$$(55)\quad \partial^2 \mathrm{CGF}_K(t)/\partial t^2 |_{t=0} = \partial^2 \mathrm{CGF}_{\mathrm{median}}(t)/\partial t^2 |_{t=0} + \sum_{i=n-K+1}^{(n-1)/2} (b^2/i^2) - \sum_{i=(n+1)/2}^{K-1} (b^2/i^2) \quad \text{for } K > \text{median}.$$

Using (13), the second cumulant can be written as

$$(56)\quad 2b^2[(\pi^2/6) - \sum_{i=1}^{(n-1)/2} i^{-2}] + \sum_{i=n-k+1}^{(n-1)/2} (b^2/i^2) - \sum_{i=(n+1)/2}^{K-1} (b^2/i^2).$$

Formula (56) checks with the formulas obtained by Plackett [11], and the numerical results attained by Birnbaum and Dudman [3].

Having convenient formulas for the covariance of any two logistic order statistics as well as their variances, it is possible by inverting the variance-covariance matrix to find best linear unbiased estimates of the logistic location parameter.

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