SMALL SAMPLE POWER OF THE ONE SAMPLE WILCOXON TEST FOR NON-NORMAL SHIFT ALTERNATIVES[1]

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0. Summary. The power of the one sample Wilcoxon test is computed for the hypothesis that the median is zero against various shift alternatives for samples drawn from several different non-normal distributions. A recursive scheme given by Klotz [2] simplifies the problem of power computations and allows investigating samples to size $n = 10$ on a large computer. The power for selected type I errors α are compared with the power of a best signed-rank procedure obtained by sorting the probabilities in decreasing order for all possible sample configurations for fixed n and adding up the probabilities associated with the most probable 100 α% of the configurations. The non-normal distributions selected for study are the t distribution with degrees of freedom $\frac{1}{2}$, 1, 2, and 4. The one sample Wilcoxon test, found to be powerful for normal shift alternatives by Klotz [2], deteriorates badly in power for the long-tailed distributions studied as does the one sample t test. However, the Wilcoxon test remains more powerful than the t test. The sign test is still more powerful than either Wilcoxon or t. No asymptotic results are obtained.

1. Introduction. Let $x_1, x_2, \cdots, x_n$ be a sample of size n from a population with probability density function f, symmetric with median μ. The one sample Wilcoxon test [5] for the hypothesis $\mu = 0$ against shift alternative $\mu \neq 0$ makes use of the statistic

$$W_+ = \sum_{k=1}^{n} kZ_{nk}$$

where $Z_{ni} = 1$ if the ith smallest observation in magnitude is non negative and 0 if the ith smallest observation in magnitude is negative. If $\mu > 0$, then large values of W_+ are likely. The power of the test for type I error α is computed by finding the $2^n\alpha$ orderings $Z_n = (Z_{n1}, Z_{n2}, \cdots, Z_{nn})$ which give rise to the largest W_+ values and adding the probabilities associated with these orderings. For example, for samples of size 5, the five orderings with the largest values of W_+ corresponding to $\alpha = \frac{5}{32} = .1563$ are: (1,1,1,1,1), $W_+ = 15$; (0,1,1,1,1), $W_+ = 14$; (1,0,1,1,1), $W_+ = 13$: (1,1,0,1,1), $W_+ = 12$; (0,0,1,1,1), $W_+ = 12$.

2. Power calculations. The notation and procedure for computing the necessary probabilities follows closely that used by Klotz [2]. The probability of obtaining an ordering z_n is:

$$P(Z_n = z_n) = n! \int \cdots \int_{0<t_1<\cdots<t_n<\infty} \prod_{i=1}^{n} f_0(t_i - s_i\mu)\, dt_i,$$

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where $s_i = 2z_{ni} - 1$ is the sign of the variable which is ith smallest in magnitude, f_0 is the density function of the underlying distribution (assumed to be symmetric) with median $\mu = 0$. Klotz simplified the problem of integration over n dimensions by reducing the problem to one of evaluating n successive one dimensional indefinite integrals. This was done by adapting a two sample scheme of Hodges (Klotz [1], p. 502) to the one sample problem.

Klotz defines

$$A_{(z_n)}(u) = P(Z_n = z_n \quad \text{and} \quad |X_i| \leqq u \quad \text{for all} \quad i).$$

The required probabilities are $A_{(z_n)}(\infty)$ for each ordering vector z_n.

Note that

$$\begin{aligned} A_{(1)}(u) &= P(Z_1 = (1) \quad \text{and} \quad -u \leqq X_1 \leqq u) \\ &= \int_0^u f(t)\, dt. \end{aligned}$$

If the density function of the underlying distribution with median $\mu = 0$ is denoted by f_0 then,

$$\begin{aligned} A_{(1)}(u) &= \int_{-\mu}^{u-\mu} f(y + \mu)\, dy = \int_{-\mu}^{u-\mu} f_0(y)\, dy \\ &= F_0(u - \mu) - F_0(-\mu), \end{aligned}$$

where F_0 is the cdf with median $\mu = 0$. Similarly $A_{(0)}(u) = F_0(u + \mu) - F_0(\mu)$. Now define

$$(z_n, 1) = (z_{n1}, z_{n2}, \cdots, z_{nn}, 1)$$

and

$$(z_n, 0) = (z_{n1}, z_{n2}, \cdots, z_{nn}, 0).$$

The recursion relations

$$A_{(z_n,1)}(x) = (n + 1) \int_0^x A_{z_n}(u) f_0(u - \mu)\, du$$

and

$$A_{(z_n,0)}(x) = (n + 1) \int_0^x A_{z_n}(u) f_0(u + \mu)\, du$$

make it possible to generate the probabilities for z_{n+j} from z_n. Klotz obtained these recursion relations and computed the probabilities for all orderings z_n, $1 \leqq n \leqq 10$, for normal shift alternatives $\mu = 0(.25)1.5(.5)3.0$. He made use of Simpson's rule for the numerical integration with an accuracy believed to be 4 decimal places.

For non-normal distributions having long tails it is desirable to use a numerical integration procedure which permits a longer step-size than that necessary by Simpson's rule. Choosing a seven point quadrature formula (Milne [3], page 123) speeds up the computation by a factor of 3 or 4. Starting values for this formula are obtained by using the trapezoidal rule and several applications of Simpson's rule. The step size can be increased by a factor of 10 using this formula compared to Simpson's rule only. The probabilities for all orderings z_n with $1 \leqq n \leqq 10$ for shifts $\mu = .25, .50, 1.0, 2.0$ and 3.0 have been computed for the non central t distribution with $\frac{1}{2}$, 1, 2 and 4 degrees of freedom. The density functions were first scaled so that $\int_{1.645}^{\infty} f_0(t)\, dt = .05$. The accuracy of the probabilities is believed to be at least 4 places. This was achieved by varying the step-size as the integration proceeded. A typical integration makes use of steps in x of the sort 0.0(.02)1.0(.2)11.0(2)111(20)1111, etc. A satisfactory termination point of the integration depends of course on the nature of the underlying density function. For example when f_0 is a t distribution with one degree of freedom it is necessary to integrate to about $x = 10^6$.

3. Numerical results. Summary results of the power calculations are given in Tables 1, 2, 3 and 4. The columns headed W are the power for selected type I errors α for $n = 5(1)10$ for the one sample Wilcoxon test. For comparison, the columns headed L give the power of a best signed-rank procedure. These figures are obtained by sorting the probabilities in decreasing order for all orderings z_n of fixed sample size n and adding up the first $100\,\alpha\%$. It is clear from the tables that for the longer tailed distributions, the power of the Wilcoxon test (W) is rather poor relative to the best signed-rank procedure (L). Table 5 illustrates for $n = 5$, $\mu = 1.0$ the gradual weakening in power of the one sample Wilcoxon rank sum procedure as the underlying distribution becomes farther removed from the normal distribution. Examination of Table 5 for the Cauchy distribution (t with one degree of freedom) reveals a rather interesting result. Consider any ordering

$$z_n = (z_{n1}, z_{n2}, \cdots, z_{nn}).$$

Define

$$z_n' = (z_{n1}', z_{n2}', \cdots, z_{nn}'),$$

where $z_{ni}' = z_{n,n-i+1}$ for all i, $1 \leqq i \leqq n$. z_n' is the reverse ordering of z_n and vice versa. For example, the orderings (1,0,1,1,0) and (0,1,1,0,1) are reverse orderings. For an underlying Cauchy distribution $P(Z_n = z_n) = P(Z_n = z_n')$. This result follows directly from the fact that if X has a Cauchy distribution with median μ then $Y = (1 + \mu^2)/X$ has a Cauchy distribution with median μ. (This was pointed out to the author by Dr. H. F. Trotter.) This result explains the weakness of the Wilcoxon test for samples from a Cauchy distribution because reverse orderings often have very different rank sums.

Table 6 lists the leading (in the sense of rank sum) orderings for $n = 10$ and $\mu = 1.0$. It should be noticed that in the case of the normal distribution, the probabilities decrease with decreasing rank sum for approximately the first 2.5% of the orderings. For the next few orderings, the switching of probabilities from their decreasing order is only minor. However, looking at the t with 4 degrees of freedom one observes earlier switching (in the 7th largest rank sum, i.e., RS $= 49$) and substantial switching after the first 2.5% of the orderings. This lack of order in the probabilities becomes more pronounced as one moves to fewer degrees of freedom in the t distribution. In fact there is almost a complete breakdown in order for the cases of 1 and $\frac{1}{2}$ degrees of freedom. For t with $\frac{1}{2}$ degrees of freedom, a pair of reversed orderings z_n and z_n' usually have probabilities related as follows: $P(Z_n = z_n) > P(Z_n = z_n')$ provided the rank sum of z_n is less than the rank sum of z_n'. There are a few pairs of reversed orderings where this is not the case. These are cases for which the rank sums for the two orderings are almost equal, the two orderings being almost identical.

4. Power of sign test. Table 7 presents the probabilities needed for computing the power of the sign test. Table 8 provides some rough computations which allow a comparison of the Wilcoxon and the sign test for the case of the Cauchy distribution. The sign test powers were obtained by graphical interpolation (second digit not guaranteed). Because the possible α levels for the two tests do not match, the exact power of the sign test does not provide a ready comparison. Table 8 demonstrates that the sign test and Wilcoxon are comparable in power for small α but for $\alpha > .05$ and $\mu \leqq 1$ the sign test is considerably more powerful than the Wilcoxon. This is to be expected in lieu of the reversed ordering discussion of the previous section.

5. Power of the t-test for the Cauchy distribution. Table 8 lists the power of the one sample t test for samples from the Cauchy distribution for the same significance levels as those given for the Wilcoxon test. These results were obtained by generating 40,000 samples of size $n = 10$ from a Cauchy and computing the t statistic for various shifts μ. For $n < 10$ the same samples were used, dropping the unneeded observations. The accuracy is at least .005 with a confidence of .95. Hence the second digit is often in error by one unit. Even though the Wilcoxon is low in power compared to the sign test for $\alpha > .05$, it is still more powerful than the t. This is especially true when $\mu \leqq 1.0$.

6. Rank sum of squares as a supplementary criterion. Tables 5 and 6 contain a column headed RSS (rank sum of squares). This column is included to demonstrate the ability of the rank sum of squares for discriminating among orderings with equal rank sums. The rank sum of squares has been suggested by Tukey [4] as a supplementary criterion to a rank sum test for breaking ties. Tables 5 and 6 demonstrate that the rank sums of squares for fixed rank sum decrease with the probabilities in the case of an underlying normal distribution. For the other distributions the rank sum of squares fails in the same way that the rank sum fails.

7. Conclusions. It is clear that the one sample Wilcoxon rank sum test, although powerful for normal shift alternatives, (Klotz [2]), deteriorates badly in power for the long tailed distributions considered as does the one sample t test. However it still remains a more powerful test than does the one sample t test. The sign test for these long tailed distributions is more powerful than the Wilcoxon or t test.

Preliminary investigation of the one sample Wilcoxon ignoring one or two observations with the largest ranks shows promise of improving the power. Further study along these lines is suggested by the fact that the user of the Wilcoxon test might at first reject (as outliers) the very large (in magnitude) observations that often arise if the underlying distribution is as long tailed as those studied here. However, Table 8 demonstrates that the Wilcoxon test is not as sensitive to outliers as the t test is.

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TABLE 1

Selected values of power of the one sample Wilcoxon (W) for the t-distribution with ½ df compared to best signed rank procedure (L)

N	α	$\mu = .25$		$\mu = .50$		$\mu = 1.0$		$\mu = 2.0$		$\mu = 3.0$	
		L	W	L	W	L	W	L	W	L	W
5	.03125 = 1/32	.5035	.5035	.6213	.6213	.7176	.7176	.7926	.7926	.8279	.8279
	.06250 = 2/32	.6928	.5876	.7951	.6962	.8649	.7800	.9109	.8423	.9300	.8706
	.09375 = 3/32	.7769	.6098	.8700	.7103	.9273	.7885	.9606	.8473	.9727	.8741
6	.01563 = 1/64	.4390	.4390	.5649	.5649	.6715	.6715	.7566	.7566	.7972	.7972
	.03125 = 2/64	.6247	.5224	.7456	.6434	.8311	.7395	.8886	.8122	.9126	.8456
	.04688 = 3/64	.7081	.5451	.8241	.6593	.8991	.7498	.9442	.8185	.9610	.8502
	.07813 = 5/64	.8127	.5658	.8981	.6714	.9465	.7564	.9725	.8220	.9815	.8526
7	.00781 = 1/128	.3827	.3827	.5136	.5136	.6284	.6284	.7223	.7223	.7676	.7676
	.02344 = 3/128	.6415	.4870	.7770	.6116	.8690	.7127	.9261	.7907	.9480	.8268
	.05469 = 7/128	.8137	.5211	.9069	.6318	.9564	.7237	.9805	.7963	.9880	.8307
	.10938 = 14/128	.9065	.6199	.9609	.7012	.9847	.7680	.9943	.8228	.9968	.8498
8	.00781 = 2/256	.5012	.4109	.6492	.5477	.7623	.6632	.8420	.7543	.8760	.7969
	.02734 = 7/256	.7592	.4659	.8739	.5870	.9394	.6889	.9723	.7699	.9828	.8084
	.05469 = 14/256	.8599	.5111	.9375	.6138	.9745	.7030	.9901	.7767	.9944	.8127
	.09766 = 25/256	.9314	.7692	.9755	.8643	.9920	.9236	.9975	.9583	.9988	.9711
9	.00977 = 5/512	.6385	.4076	.7863	.5401	.8814	.6529	.9369	.7430	.9570	.7858
	.02734 = 14/512	.8127	.4446	.9124	.5610	.9630	.6633	.9853	.7477	.9916	.7887
	.04883 = 25/512	.8926	.5523	.9590	.6461	.9859	.7226	.9956	.7855	.9978	.8168
	.10156 = 52/512	.9577	.7883	.9877	.8808	.9968	.9358	.9993	.9664	.9997	.9772
10	.00977 = 10/1024	.7110	.3842	.8502	.5137	.9284	.6276	.9681	.7212	.9807	.7665
	.02441 = 25/1024	.8487	.4450	.9389	.5530	.9779	.6497	.9927	.7325	.9963	.7739
	.05273 = 54/1024	.9321	.7293	.9789	.8444	.9942	.9153	.9985	.9555	.9994	.9698
	.09668 = 99/1024	.9696	.7729	.9925	.8681	.9984	.9269	.9997	.9608	.9999	.9731

TABLE 2

Selected values of power of the one sample Wilcoxon (W) for Cauchy distribution compared to best signed rank procedure (L)

N	α	$\mu = .25$		$\mu = .50$		$\mu = 1.0$		$\mu = 2.0$		$\mu = 3.0$	
		L	W	L	W	L	W	L	W	L	W
5	.03125	.2264	.2264	.4351	.4351	.6542	.6542	.8096	.8096	.8692	.8692
	.06250	.3326	.3326	.5608	.5608	.7612	.7612	.8816	.8816	.9225	.9225
	.09375	.4389	.3959	.6864	.6132	.8681	.7902	.9535	.8938	.9759	.9294
6	.01563	.1682	.1682	.3684	.3684	.6010	.6010	.7761	.7761	.8451	.8451
	.03125	.2527	.2527	.4860	.4860	.7125	.7125	.8560	.8560	.9057	.9057
	.04688	.3372	.3038	.6035	.5365	.8241	.7444	.9358	.8707	.9663	.9142
	.07813	.4393	.3716	.7046	.5903	.8879	.7713	.9652	.8805	.9833	.9192
7	.00781	.1249	.1249	.3120	.3120	.5521	.5521	.7440	.7440	.8217	.8217
	.02344	.2578	.2324	.5281	.4683	.7797	.7004	.9169	.8476	.9558	.8990
	.05469	.4047	.3313	.6896	.5559	.8869	.7466	.9667	.8648	.9845	.9078
	.10938	.5705	.4600	.8308	.6648	.9583	.8071	.9923	.8904	.9973	.9221
8	.00781	.1446	.1446	.3622	.3622	.6214	.6214	.8051	.8051	.8718	.8718
	.02734	.3142	.2557	.6134	.4893	.8501	.7060	.9542	.8438	.9784	.8938
	.05469	.4547	.3455	.7582	.5666	.9336	.7449	.9866	.8577	.9952	.9007
	.09766	.5872	.4990	.8546	.7531	.9702	.9106	.9957	.9726	.9987	.9868
9	.00977	.2007	.1702	.4846	.4007	.7686	.6503	.9200	.8168	.9601	.8768
	.02734	.3545	.2644	.6808	.4943	.9026	.7003	.9786	.8345	.9919	.8854
	.04883	.4693	.3589	.7849	.5944	.9495	.7669	.9916	.8665	.9973	.9044
	.10156	.6322	.5330	.8916	.7901	.9835	.9340	.9983	.9827	.9996	.9924
10	.00977	.2282	.1768	.5440	.4067	.8261	.6470	.9512	.8089	.9785	.8691
	.02441	.3725	.2686	.7161	.5026	.9265	.7014	.9869	.8297	.9956	.8799
	.05273	.5240	.4248	.8387	.7160	.9721	.9043	.9969	.9743	.9992	.9886
	.09668	.6593	.5416	.9137	.7975	.9897	.9352	.9992	.9821	.9998	.9917

TABLE 3

Selected values of power of the one sample Wilcoxon (W) for the t-distribution with 2 df compared to best signed rank procedure (L)

N	α	$\mu = .25$		$\mu = .50$		$\mu = 1.0$		$\mu = 2.0$		$\mu = 3.0$	
		L	W	L	W	L	W	L	W	L	W
5	.03125	.1154	.1154	.2624	.2624	.5603	.5603	.8338	.8338	.9186	.9186
	.06250	.1996	.1996	.3994	.3994	.7162	.7162	.9224	.9224	.9681	.9681
	.09375	.2646	.2646	.4816	.4816	.7767	.7767	.9513	.9410	.9837	.9750
6	.01563	.0749	.0749	.2008	.2008	.4990	.4990	.8041	.8041	.9031	.9031
	.03125	.1316	.1316	.3127	.3127	.6530	.6530	.9022	.9022	.9599	.9599
	.04688	.1763	.1763	.3823	.3823	.7162	.7162	.9336	.9242	.9774	.9683
	.07813	.2519	.2483	.4889	.4749	.8100	.7787	.9654	.9398	.9899	.9733
7	.00781	.0486	.0486	.1536	.1536	.4444	.4444	.7754	.7754	.8879	.8879
	.02344	.1169	.1169	.3019	.3019	.6581	.6581	.9151	.9069	.9707	.9614
	.05469	.2146	.2093	.4597	.4398	.8075	.7636	.9691	.9346	.9917	.9705
	.10938	.3475	.3296	.6298	.5819	.9150	.8473	.9935	.9557	.9989	.9782
8	.00781	.0567	.0567	.1896	.1896	.5386	.5386	.8606	.8606	.9425	.9425
	.02734	.1435	.1407	.3675	.3535	.7498	.7109	.9566	.9206	.9874	.9647
	.05469	.2362	.2227	.5157	.4696	.8693	.7875	.9886	.9384	.9980	.9706
	.09766	.3450	.3324	.6456	.6155	.9309	.9013	.9962	.9886	.9995	.9973
9	.00977	.0742	.0742	.2434	.2413	.6375	.6190	.9216	.8948	.9750	.9552
	.02734	.1589	.1504	.4173	.3795	.8200	.7351	.9823	.9241	.9969	.9646
	.04883	.2367	.2225	.5359	.4870	.8928	.8116	.9932	.9465	.9991	.9734
	.10156	.3749	.3599	.6934	.6607	.9550	.9305	.9986	.9945	.9999	.9990
10	.00977	.0826	.0805	.2788	.2646	.7019	.6506	.9525	.9021	.9876	.9562
	.02441	.1608	.1500	.4394	.3920	.8505	.7525	.9892	.9278	.9985	.9650
	.05273	.2661	.2538	.5918	.5562	.9262	.8929	.9972	.9905	.9997	.9982
	.09668	.3857	.3671	.7210	.6810	.9665	.9416	.9994	.9955	1.0000	.9991

TABLE 4

Selected values of power of the one sample Wilcoxon (W) for the t-distribution with 4 df compared to best signed rank procedure (L)

N	α	$\mu = .25$		$\mu = .50$		$\mu = 1.0$		$\mu = 2.0$		$\mu = 3.0$	
		L	W	L	W	L	W	L	W	L	W
5	.03125	.0905	.0905	.1979	.1979	.4899	.4899	.8564	.8564	.9560	.9560
	.06250	.1650	.1650	.3270	.3270	.6744	.6744	.9516	.9516	.9909	.9909
	.09375	.2270	.2270	.4154	.4154	.7576	.7576	.9708	.9708	.9946	.9946
6	.01563	.0560	.0560	.1431	.1431	.4248	.4248	.8302	.8302	.9475	.9475
	.03125	.1032	.1032	.2412	.2412	.5999	.5999	.9363	.9363	.9880	.9880
	.04688	.1434	.1434	.3111	.3111	.6842	.6842	.9597	.9597	.9927	.9927
	.07813	.2126	.2126	.4139	.4139	.7759	.7759	.9767	.9755	.9967	.9951
7	.00781	.0346	.0346	.1035	.1035	.3683	.3683	.8049	.8049	.9390	.9390
	.02344	.0902	.0902	.2314	.2314	.6146	.6146	.9477	.9477	.9906	.9906
	.05469	.1748	.1748	.3767	.3767	.7654	.7654	.9813	.9755	.9974	.9950
	.10938	.2913	.2897	.5369	.5304	.8862	.8684	.9969	.9883	.9999	.9972
8	.00781	.0401	.0401	.1298	.1298	.4697	.4697	.9037	.9037	.9816	.9816
	.02734	.1114	.1114	.2861	.2861	.7014	.7014	.9731	.9677	.9961	.9936
	.05469	.1875	.1871	.4152	.4108	.8270	.8073	.9939	.9815	.9997	.9957
	.09766	.2865	.2854	.5509	.5469	.9076	.8994	.9984	.9969	1.0000	.9997
9	.00977	.0539	.0539	.1761	.1761	.5815	.5815	.9476	.9476	.9917	.9904
	.02734	.1202	.1201	.3167	.3157	.7640	.7472	.9897	.9749	.9995	.9945
	.04883	.1859	.1845	.4316	.4238	.8572	.8339	.9970	.9864	.9999	.9967
	.10156	.3109	.3097	.5967	.5925	.9352	.9301	.9994	.9989	1.0000	.9999
10	.00977	.0591	.0591	.1994	.1994	.6373	.6366	.9700	.9605	.9967	.9921
	.02441	.1195	.1189	.3330	.3274	.8010	.7749	.9949	.9795	.9998	.9952
	.05273	.2097	.2087	.4835	.4789	.8988	.8902	.9986	.9978	1.0000	.9999
	.09668	.3163	.3150	.6186	.6138	.9513	.9449	.9996	.9993	1.0000	.9999

TABLE 5

Comparison of probabilities for several distributions for $N = 5$ and $\mu = 1.0$

Binary	Octal	RS	RSS	Normal*	$t_{4\,df}$	$t_{2\,df}$	Cauchy	$t_{\frac{1}{2}\,df}$
11111	= 37	15	55	.4216	.4900	.5603	.6542	.7176
01111	= 17	14	54	.2070	.1845	.1560	.1070	.0624
10111	= 27	13	51	.1047	.0832	.0605	.0291	.0086
00111	= 07	12	50	.0632	.0408	.0239	.0087	.0024
11011	= 33	12	46	.0525	.0456	.0350	.0177	0050
01011	= 13	11	45	.0307	.0201	.0113	.0033	.0005
11101	= 35	11	39	.0244	.0311	.0324	.0291	.0229
10011	= 23	10	42	.0187	.0104	.0050	.0011	.0002
01101	= 15	10	38	.0139	.0125	.0090	.0042	.0014
11110	= 36	10	30	.0089	.0314	.0603	.1070	.1473
00011	= 03	9	41	.0126	.0059	.0024	.0004	.0001
10101	= 25	9	35	.0082	.0060	.0036	.0012	.0002
01110	= 16	9	29	.0049	.0118	.0158	.0157	.0110
00101	= 05	8	34	.0054	.0031	.0015	.0003	.0004
11001	= 31	8	30	.0049	.0034	.0022	.0011	.0009
10110	= 26	8	26	.0028	.0053	.0060	.0042	.0014

* Results obtained from Dr. Jerome Klotz and verified by the author.

TABLE 6

Comparison of probabilities for several distributions for $N = 10$ and $\mu = 1.0$

Binary	Octal	RS	RSS	Normal*	t_{4df}	t_{2df}	Cauchy	$t_{\frac{1}{2}df}$
1 111 111 111 =	1777	55	385	.1777	.2401	.3139	.4280	.5149
0 111 111 111 =	0777	54	384	.1137	.1242	.1269	.1113	.0785
1 011 111 111 =	1377	53	381	.0750	.0711	.0616	.0391	.0168
0 011 111 111 =	0377	52	380	.0531	.0434	.0322	.0163	.0061
1 101 111 111 =	1577	52	376	.0505	.0447	.0357	.0185	.0050
0 101 111 111 =	0577	51	375	.0353	.0263	.0172	.0066	.0015
1 110 111 111 =	1677	51	369	.0344	.0305	.0240	.0115	.0022
1 001 111 111 =	1177	50	372	.0254	.0167	.0097	.0030	.0006
0 110 111 111 =	0677	50	368	.0238	.0173	.0109	.0036	.0005
1 111 011 111 =	1737	50	360	.0235	.0223	.0185	.0092	.0015
0 001 111 111 =	0177	49	371	.0191	.0113	.0058	.0016	.0003
1 010 111 111 =	1277	49	365	.0169	.0108	.0058	.0015	.0002
0 111 011 111 =	0737	49	359	.0161	.0124	.0080	.0025	.0003
1 111 101 111 =	1757	49	349	.0158	.0175	.0159	.0092	.0020
0 010 111 111 =	0277	48	364	.0127	.0071	.0033	.0007	.00008
1 100 111 111 =	1477	48	360	.0123	.0072	.0036	.0008	.00007
1 011 011 111 =	1337	48	356	.0114	.0075	.0041	.0010	.00007
0 111 101 111 =	0757	48	348	.0108	.0095	.0066	.0024	.00027
1 111 110 111 =	1767	48	336	.0104	.0145	.0154	.0115	.0044
0 100 111 111 =	0477	47	359	.0091	.0046	.0020	.0003	.00003
0 011 011 111 =	0337	47	355	.0084	.0048	.0023	.0004	.00003
1 101 011 111 =	1537	47	351	.0082	.0049	.0024	.0005	.00003
1 011 101 111 =	1357	47	345	.0075	.0056	.0033	.0008	.00006
0 111 110 111 =	0767	47	335	.0070	.0077	.0062	.0028	.00056
1 111 111 011 =	1773	47	321	.0065	.0130	.0171	.0185	.0138
1 000 111 111 =	1077	46	356	.0069	.0032	.0012	.0002	.00002
0 101 011 111 =	0537	46	350	.0060	.0031	.0013	.0002	.00001
1 110 011 111 =	1637	46	344	.0060	.0035	.0017	.0003	.00001
0 011 101 111 =	0357	46	344	.0056	.0036	.0018	.0004	.00002
1 101 101 111 =	1557	46	340	.0054	.0036	.0019	.0004	.00002
1 011 110 111 =	1367	46	332	.0049	.0045	.0030	.0010	.0001
0 111 111 011 =	0773	46	320	.0044	.0068	.0068	.0044	.0018
1 111 111 101 =	1775	46	304	.0037	.0130	.0232	.0391	.0498
0 000 111 111 =	0077	45	355	.0054	.0023	.0008	.0001	.00001
1 001 011 111 =	1137	45	347	.0045	.0021	.0008	.0001	.00000
0 110 011 111 =	0637	45	343	.0044	.0021	.0008	.0001	.00000
0 101 101 111 =	0557	45	339	.0039	.0022	.0010	.0001	.00001

TABLE 6—*Continued*

Binary	Octal	RS	RSS	Normal*	t_{4df}	t_{2df}	Cauchy	$t_{\frac{1}{2}df}$
1 110 101 111	= 1657	45	333	.0039	.0025	.0013	.0003	.00001
0 011 110 111	= 0367	45	331	.0036	.0028	.0016	.0004	.00003
1 101 110 111	= 1567	45	327	.0035	.0029	.0018	.0005	.00003
1 011 111 011	= 1373	45	317	.0030	.0039	.0033	.0015	.00032
0 111 111 101	= 0775	45	303	.0024	.0067	.0091	.0095	.0067
1 111 111 110	= 1776	45	285	.0016	.0173	.0473	.1113	.1793
0 001 011 111	= 0137	44	346	.0035	.0015	.0005	.0001	.00000
1 010 011 111	= 1237	44	340	.0033	.0014	.0005	.0001	.00000
1 001 101 111	= 1157	44	336	.0030	.0015	.0006	.0001	.00000
0 110 101 111	= 0657	44	332	.0028	.0015	.0006	.0001	.00000
0 101 110 111	= 0567	44	326	.0025	.0017	.0009	.0002	.00001
1 111 001 111	= 1717	44	324	.0028	.0019	.0010	.0002	.00001
1 110 110 111	= 1667	44	320	.0025	.0020	.0012	.0003	.00002
0 011 111 011	= 0373	44	316	.0022	.0024	.0017	.0006	.00011
1 101 111 011	= 1573	44	312	.0021	.0025	.0019	.0007	.00009
1 011 111 101	= 1375	44	300	.0017	.0038	.0044	.0032	.0013
0 111 111 110	= 0776	44	284	.0011	.0089	.0186	.0278	.0258

* Results obtained from Dr. Jerome Klotz.

TABLE 7

Probability that an observation is positive when sampled from an underlying t distribution (adjusted for scale)

Degrees of freedom	Shift μ				
	.25	.50	1.0	2.0	3.0
$\frac{1}{2}$	.8718	.9092	.9358	.9546	.9629
1	.7429	.8467	.9186	.9586	.9723
2	.6493	.7652	.8906	.9643	.9832
4	.6186	.7233	.8670	.9695	.9910
∞	.5987	.6915	.8413	.9772	.9987

TABLE 8

Comparison of power of Sign (S), Wilcoxon (W) and t-tests (T) for samples from Cauchy distribution with median μ

N	α	μ = .25			μ = .50			μ = 1.0			μ = 2.0			μ = 3.0		
		S	W	T	S	W	T	S	W	T	S	W	T	S	W	T
5	.03125	.23	.23	.14	.44	.44	.36	.65	.65	.61	.81	.81	.79	.87	.87	.86
	.06250	.34	.33	.25	.57	.56	.47	.76	.76	.69	.88	.88	.83	.93	.92	.89
	.09375	.43	.40	.33	.66	.61	.54	.83	.79	.74	.92	.89	.86	.96	.93	.91
6	.01563	.17	.17	.09	.37	.37	.28	.60	.60	.55	.78	.78	.76	.85	.85	.84
	.03125	.25	.25	.15	.49	.49	.38	.70	.71	.63	.85	.86	.80	.90	.91	.87
	.04688	.30	.30	.21	.56	.54	.44	.76	.74	.68	.90	.87	.83	.93	.91	.89
	.07813	.43	.37	.29	.69	.59	.52	.85	.77	.73	.95	.88	.87	.98	.92	.91
7	.00781	.12	.12	.06	.31	.31	.23	.55	.55	.50	.74	.74	.72	.82	.82	.82
	.02344	.23	.23	.14	.52	.47	.35	.72	.70	.61	.87	.85	.80	.91	.90	.86
	.05469	.40	.33	.25	.68	.56	.48	.87	.75	.70	.96	.86	.85	.97	.91	.89
	.10938	.55	.46	.39	.81	.66	.59	.96	.81	.78	.99	.89	.88	1.00	.92	.92
8	.00781	.15	.14	.07	.36	.36	.26	.64	.62	.53	.80	.81	.75	.87	.87	.83
	.02734	.31	.26	.16	.58	.49	.40	.83	.71	.65	.94	.84	.81	.97	.89	.87
	.05469	.43	.35	.26	.74	.57	.49	.91	.74	.71	.98	.86	.85	.99	.90	.89
	.09766	.55	.50	.36	.85	.75	.58	.95	.91	.77	.99	.97	.88	1.00	.99	.92
9	.00977	.19	.17	.09	.44	.40	.28	.75	.65	.56	.90	.82	.76	.94	.88	.84
	.02734	.33	.26	.17	.63	.49	.40	.87	.70	.64	.96	.83	.81	.98	.89	.87
	.04883	.44	.36	.25	.73	.59	.48	.92	.77	.70	.98	.87	.84	.99	.90	.89
	.10156	.60	.53	.37	.87	.79	.59	.98	.93	.77	1.00	.98	.88	1.00	.99	.92
10	.00977	.22	.18	.09	.53	.41	.30	.80	.65	.57	.93	.81	.77	.97	.87	.84
	.02441	.36	.27	.16	.68	.50	.40	.89	.70	.64	.97	.83	.81	.99	.88	.87
	.05273	.50	.42	.26	.81	.72	.50	.96	.90	.72	.99	.97	.84	1.00	.99	.90
	.09668	.64	.54	.37	.89	.80	.59	.99	.94	.77	1.00	.98	.88	1.00	.99	.92

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