BOOK REVIEWS

Correspondence concerning reviews should be addressed to the Book Review Editor, Professor Jack Kiefer, Department of Mathematics, Cornell University, Ithaca, New York 14850.

RICHARD BELLMAN (editor), *Stochastic Processes in Mathematical Physics and Engineering*. American Mathematical Society, Providence, 1964. viii + 318 pp. $7.60 ($5.70 to members).

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There are many problems in which randomness arises naturally or else is thought to be a convenient artifice to introduce because of the complexity of the context dealt with. A number of these problems come to mind readily—wave propagation in random media, information retrieved from messages corrupted by noise, optimization problems when dealing with systems involving random parameters, et al. Most of the papers in this symposium volume discuss a number of mathematically posed problems of this type that have arisen from questions in physics or engineering. Most of the papers unfortunately do not discuss the physical background of the problems though sufficient references are sometimes found to current relevant literature. An exception is to be found in the papers (of Twersky, Hoffman and Keller) on wave propagation in random media.

Very often the new problems posed are of the following character. Let us say that in an original deterministic context one is led to consider the solution of a differential or integral equation. Now in the present context the coefficients of the differential equation or the kernel of the integral equation become random processes. Questions of existence and uniqueness arise as they do in the deterministic context. But assuming these resolved, there is the much more difficult question of qualitative characterization of the solutions. In the paper of Bharucha-Reid there are some general results on "random operators" that are mentioned. However, in this generality the results intuitively seem to be basically restatements of counterparts for deterministic operators with the "randomness" scarcely influencing the character of the theorems obtained. One can scarcely expect much more without some specialization. Bharucha-Reid also refers in passing to a very interesting special class of problems in which important work has been carried on recently. Bellman's article refers to these problems in more detail—in fact, he carried out some early work in this area. Consider independent, identically distributed random $k \times k$ matrices M_j, $j = 1, 2, \cdots$. What can one say about the limiting behavior of the random product $T_n = M_n \cdots M_1$

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as $n \rightarrow \infty$? This suggests the study of limit laws and limiting behavior for products of independent (and generally noncommuting) elements of a group, semigroup or some other algebraic structure. Usually there will also be some topological assumptions imposed on the algebraic structure, for example, the group may be assumed to be compact or locally compact. Recently, some analogues of the classical results on infinitely divisible laws have been obtained when the structure is a Lie group. However, there are still many open questions yet in the case of the elements chosen from matrix groups. Recent work on this area has been surveyed in the book of U. Grenander, *Probabilities on Algebraic Structures*, Wiley, 1963. Later work in the case of groups has been carried on by H. Furstenberg (Noncommuting random products, *Trans. Amer. Math. Soc.* **108** 1963) and V. N. Tutuballin (Limit behavior of the convolution of measures on a complex unimodular group, *Soviet Math. Dok.* **3** II, 1962), among others.

The papers on wave propagation in random media and that of Richardson discuss approximate procedures in getting information on random solutions of differential equations (possibly with random coefficients). These are formal and there is unfortunately very little information about the domain in which they are valid. Keller's paper gives an excellent discussion of some of these procedures in the case of wave propagation problems. Perturbation methods (already familiar in deterministic problems) are considered as well as a method referred to as that of hierarchy equations. Here one generates a family of equations for moments of the random solution. Unfortunately, typically the equations give nth order moments in terms of higher moments so that one has an infinite family of equations. A usual procedure is to truncate the family by assuming an approximate simple relation for a high order moment or cumulant in terms of lower moments suggested by what would be true in the case of independence or a Gaussian process for that moment or cumulant. Another approach is to use a characteristic functional. Such procedures are quite familiar in the study of idealized models of turbulence (see, for example, G. K. Batchelor, *The Theory of Homogeneous Turbulence*, Cambridge, 1953, and E. Hopf, Statistical hydromechanics and functional calculus, *J. Rational Mech. Anal.* **1** (1952), 87) and it is surprising that there is only a casual reference to the work carried out there. Perhaps this is due to the fact that the study of turbulence has proven to be a very knotty problem.

Parzen has contributed an expository paper on statistical spectral analysis of time series referring to much of his work in the area. Recently there has been interest in higher order spectral analysis, that is, for moments or cumulants of order greater than two. This has been used in the analysis of some nonlinear effects in oceanography (see the paper of Hasselman, Munk and MacDonald, Bispectra of ocean waves, *Proc. Symp. Time Ser. Analysis*, John Wiley, 1963). There are many other papers in the Time Series Symposium volume that consider problems in time series analysis as they arise in a variety of applied fields such as meteorology and econometrics.

Other papers in this symposium volume consider applications of dynamic programming, stochastic approximation in an optimization problem as well as Markovian decision processes. The more detailed comments on individual papers made in this review are a reflection of the reviewer's interest. The volume contains a stimulating collection of papers that touch on many of the new and difficult problems on stochastic processes that are motivated by applications in the physical sciences and engineering.