ANOTHER CHARACTERISTIC PROPERTY OF THE CAUCHY DISTRIBUTION

By M. V. Menon[1]

IBM Research Labs., San Jose, Calif.

1. Main result. The purpose of this paper is to prove the theorem and corollary stated in this section. The corollary answers the question raised in Section 3 of [4] to which we also refer the reader for further motivation.

Theorem. *Let X be a symmetric* r.v., *and X_i, $i = 1, 2, \cdots$,* r.v.'s *independently and identically distributed as X. The two conditions:* (i) *for any real number c, and positive integer n, there exist real numbers $A = A(n, c)$ and $B = B(n, c)$ for which $\sum_1^n 1/(X_i + c)$ is distributed as $A/(X + B)$, and* (ii) *for some $c \neq 0$ the symmetric* r.v. *$1/(X_1 + c) + 1/(X_2 - c)$ is distributed as $A(c)/X$, for some number $A(c)$, are necessary and sufficient for X to have a Cauchy distribution.*

Corollary. *Let X_i, $i = 1, 2, \cdots$ be* r.v.'s *independently and identically distributed as a* r.v. *X. The necessary and sufficient condition that, for any real numbers $a_i \neq 0$, b_i, $i = 1, 2, \cdots$ and any positive integer n, there exist real numbers A and B for which $\sum_1^n 1/(a_iX_i + b_i)$ has the same distribution as $A/(X + B)$, is that X have the Cauchy distribution.*

2. Notation. We set down some of the notation used which will conform to that of [4] as far as possible. $\xi_1, \xi_2, \cdots$, denote r.v.'s independently and identically distributed as any given r.v. ξ. When two r.v.'s ξ and η are set equal to each other, $\xi = \eta$, we mean only that they have the same distribution, and *not* that they are equal with probability one. The symbols $\rightarrow_p$ and $\rightarrow_w$ stand for convergence in probability and weak convergence respectively. Slightly modifying the usual meaning of the symbol $\sim$, we write $f_n \sim g_n$, if f_n/g_n converges to a positive constant as n tends to infinity. The pdf (probability density function) and ch.f. (characteristic function) of $1/X$ will be denoted by $f(x)$ and $\phi(t)$ resp., and the pdf of X by $g(x)$. (That the pdf's mentioned exist will follow in the course of the proof.) Finally, by $\mathcal{L}(\xi)$ is meant the law of any r.v. ξ.

N.B. Only non-degenerate r.v.'s are considered in this paper.

3. Proofs. We first prove that the conditions stated in the theorem are sufficient. Therefore, let

$$\sum_1^n 1/(X_i + c) = A(n, c)/[X + B(n, c)] \tag{3.1}$$

where $A(n, c)$ is assumed, without any loss of generality, to be positive, and

$$1/(X_1 + c) + 1/(X_2 - c) = A(c)/X \tag{3.2}$$

where $c \neq 0$.

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[1] Now at Math. Res. Centre, Univ. of Wisconsin.

LEMMA 1. *$1/X$ is stable and has the* ch.f. $\phi(t) = \exp(-\mu|t|^\alpha), \mu > 0, 0 < \alpha \leqq 2$.

PROOF. Denote $1/X$ by Y. Since X and therefore Y is symmetric it follows from (3.1) that $\sum_1^n Y_i = a(n)Y$, for some function $a(n)$ of n.

Hence ([1], p. 162), Y is stable. Since it is also symmetric we have ([1], pp. 51, 164) $\phi(t) = \exp(-\mu|t|^\alpha)$, where $\mu > 0$ and $0 < \alpha \leqq 2$.

LEMMA 2. *Given any real number c and any positive integer n, there exist numbers $a(n, c)$, $b(n, c)$ such that $1/(X + c) = \sum_1^n a(n, c)/[X_i + b(n, c)]$, and hence, in particular, $1/(X + c)$ is infinitely divisible.*

PROOF. We have, as $c' \to c$

$$[\textstyle\sum_1^n 1/(X_i + c')]^{-1} \to_p [\sum_1^n 1/(X_i + c)]^{-1}.$$

Hence, from (3.1)

$$\mathcal{L}\{[X + B(n, c')]/A(n, c')\} \to_w \mathcal{L}\{[X + B(n, c)]/A(n, c)\}.$$

Therefore ([1], p. 42), $A(n, c') \to A(n, c)$ and $B(n, c') \to B(n, c)$. Thus, in particular, $B(n, c)$ is a continuous function of c.

Next, it will be shown that $B(n, c)$ is an unbounded function of c.

For any fixed n, $(c/n) \sum_1^n 1/(X_i + c) \to_p 1$, as $c \to \pm\infty$. Hence, from (3.1), as $c \to \pm\infty$,

$$\mathcal{L}\{cA(n, c)/n[X + B(n, c)]\} \to_w \epsilon(x - 1),$$

where $\epsilon(x - 1)$ is the distribution function of the r.v. which is equal to unity with probability 1. Therefore $[n/cA(n, c)][X + B(n, c)] \to_p 1$, which implies that $n/cA(n, c) \to 0$ and $nB(n, c)/cA(n, c) \to 1$. Hence $B(n, c) \to \pm\infty$ according as $c \to \pm\infty$. Since $B(n, c)$ has already been shown to be continuous in c, it follows that $B(n, c)$ takes all real values as c varies over the real line, and the lemma is established.

LEMMA 3. *$g(x)$ is continuous for every x if $\alpha \geqq 1$, and for all x except $x = 0$ if $\alpha < 1$. $g(x) \sim |x|^{-2}$ as $x \to \pm\infty$ and $g(x) \sim |x|^{\alpha-1}$ as $x \to 0$, $\alpha < 2$.*

PROOF. The proof follows from the facts that $g(x) = x^{-2}f(1/x)$, that $f(x)$ being the pdf of the stable *r.v.* $1/X$ (see Lemma 1), is continuous everywhere, and that $f(x) \sim |x|^{-\alpha-1}$ as $x \to \pm\infty$ ([5] or [4]), and $f(0) \neq 0$.

By Lemma 2, we have for any c

$$1/(X + c) = \textstyle\sum_1^n a(n, c)/[X_i + b(n, c)], \tag{3.3}$$

and since $1/(X + c)$ is infinitely divisible, the Lévy-Khinchin representation ([1], p. 76)

$$\log \psi(t) = i\gamma(c)t + \int (e^{itu} - 1 - itu/1 + u^2)(1 + u^2/u^2)\, d\theta(u). \tag{3.4}$$

Let $g_n(x)$ be the pdf of $a(n, c)/[X + b(n, c)]$. Then

$$g_n(x) = a(n, c)g[a(n, c)/x - b(n, c)]/x^2. \tag{3.5}$$

By ([1], p. 76 ff.) and (3.3) we have

$$\textstyle\int_{-\infty}^{u} (1/1 + x^2)g_n^*(x)\, dx \to_w \theta(u), \tag{3.6}$$

where $g_n^*(x) = nx^2g_n(x)$.

We proceed to consider the various limiting forms as $n \to \infty$, of the expression $g_n^*(x)$ that occurs in the integrand of (3.6), under different assumptions about the limiting behaviour of $a(n, c)/b(n, c)$. For convenience we shall write a_n and b_n for $a(n, c)$ and $b(n, c)$. It will be understood that whenever an expression such as $a_n/b_n \to k$ occurs, it is meant only that k is a limit point of $\{a_n/b_n\}$ and that the convergence indicated is for some appropriate sub-sequence. Further, when we write for example (see (a) below) $a_n/b_n \to k$, $a_n \to k'$, $|b_n| \to k''$, we mean only that there exists a sub-sequence $\{n_i\}$ of the positive integers, such that $a_{n_i}/b_{n_i} \to k$, $a_{n_i} \to k'$, and $|b_{n_i}| \to k''$, as $i \to \infty$.

Let $k \neq 0$. From (3.5), taking into account Lemma 3, we see that the following cases arise:

(a) $a_n/b_n \to k$, $a_n \to k'$, $|b_n| \to k''$, and $g_n^*(x) \sim nk'g(k'/x - k'')$, where $k' \neq 0$, $k'' \neq 0$, $k = k'/k''$.

(b) $a_n/b_n \to k$, $a_n \to \infty$, $|b_n| \to \infty$, and $g_n^*(x) \sim na_n|b_n|^{-2}|k/x - 1|^{-2}$

(c) $a_n/b_n \to k$, $a_n \to 0$, $b_n \to 0$, and $g_n^*(x) \sim na_n|b_n|^{\alpha-1}|k/x - 1|^{\alpha-1}$, $\alpha < 2$

(d) $a_n/b_n \to 0$, $a_n \to 0$, $b_n \to k'$, and $g_n^*(x) \sim na_n|b_n|^{\alpha-1}$, where $k' \geqq 0$, $\alpha < 2$, or $k' \neq 0$, $\alpha = 2$.

(e) $a_n/b_n \to 0$, $a_n \to 0$, $|b_n| \to \infty$, and $g_n^*(x) \sim na_n|b_n|^{-2}$

(f) $a_n/b_n \to 0$, $a_n \to \infty$, $|b_n| \to \infty$, and $g_n^*(x) \sim na_n|b_n|^{-2}$

(g) $b_n/a_n \to 0$, $a_n \to 0$, $b_n \to 0$, and $g_n^*(x) \sim na_n^{\alpha}|x|^{1-\alpha}$, $\alpha < 2$.

(h) $b_n/a_n \to 0$, $a_n \to k'$, $b_n \to 0$, and $g_n^*(x) \sim nk'g(k'/x)$, where $k' \neq 0$.

(k) $b_n/a_n \to 0$, $a_n \to \infty$, $|b_n| \to \infty$, and $g_n^*(x) \sim na_n^{-1}|x|^{+2}$

(m) $a_n \to 0$, $b_n \to 0$, $\alpha = 2$.

It is now clear from (3.6), using the lemma of Fatou, that since $\theta(u)$ is bounded ([1], p. 76), Cases (a) and (h) may be ignored and it may be assumed that $na_n|b_n|^{-2}$ in Cases (b), (e), and (f), $na_n|b_n|^{\alpha-1}$ in (c) and (d), na_n^{-1} in (k), and na_n^{α} in (g), converge. Let us indicate (generically) this limit of convergence by L. We list below the effect on $\theta(u)$ and hence on $1/(X + c)$ of the various cases other than (a) and (h), holding.

We shall denote, for any u, v, $u \leqq v$, the integrals $\int_u^v [1/(1 + x^2)]g_n^*(x)\, dx$ and $\int_u^v [1/(1 + x^2)] \lim_{n\to\infty} g_n^*(x)\, dx$ by $I_n(u, v)$ and $I(u, v)$ respectively. ϵ will denote a positive number. c_n will stand for a_n/b_n in Cases (b) to (f), and for b_n/a_n in the remaining cases. Since the proofs are easy, the lines of proof are merely indicated. In them, use is made in an obvious manner of the following facts:

(i) $f(x) < M|x|^{-\alpha-1}$, where M is a positive constant.

(ii) $f(x) > N|x|^{-\alpha-1}$, for all x sufficiently large, where N is a positive constant, and $\alpha < 2$.

(iii) The pdf of $1/(X + c)$ is either zero or infinite at $x = 1/c$.

(iv) $g_n^*(x)$ can be written as $na_n|b_n|^{-2}(c_n/x - 1)^{-2}f(1/b_n)c_n/x - 1))$, where $c_n = a_n/b_n$, or as $na_n^{-1}(1/x - c_n)^{-2}f(1/a_n(1/x - c_n))$, where $c_n = b_n/a_n$.

(i) and (ii) are clear consequences of the fact, previously mentioned, that $f(x) \sim |x|^{-\alpha-1}$, and (iii) follows from Lemma (3).

Case (b). $I_n(-\infty, k - \epsilon) \to I(-\infty, k - \epsilon)$ and $I_n(k + \epsilon, u) \to I(k + \epsilon, u)$, by the Lebesgue dominated convergence theorem. (Lebesgue).

If $L \neq 0$, $I(-\infty, k - \epsilon) \to \infty$ as $\epsilon \to 0$, which means that $\theta(u)$ is unbounded, an inadmissible situation.

If $L = 0$, $\theta(u)$ is a step-function with the step at $u = k$. If the step-size is zero, then $\theta(u) \equiv 0$ and hence $1/(X + c)$ is an improper r.v. If the step-size is non-zero, then $1/(X + c)$ has a Poisson distribution, and this is clearly impossible.

Case (c). If $L \neq 0$ and $\alpha < 2$, then $I_n(u, v) \to I(u, v)$, by Lebesgue, for any interval (u, v) for which $u \leqq k \leqq v$ does not hold. By direct estimation, one shows that $I_n(k - \epsilon, k + \epsilon) \to 0$ as $\epsilon \to 0$, uniformly with respect to n. Hence $\theta(u) = I(-\infty, u)$.

Finally, $L = 0$, $\alpha < 2$ also leads to an inadmissible situation similar to that in Case (b) above.

Case (d). If $L \neq 0$ and $\alpha = 2$, one shows that $I_n(-\infty, u) \to I(-\infty, u)$.

If $L \neq 0$, $\alpha < 2$, then, just as in Case (c), $\theta(u) = I(-\infty, u)$.

Next, $L = 0$ is inadmissible for the same reason as in Case (c).

Case (e). Let $c_n < 0$, $A > 1$, and $\lambda_n = b_n/(b_n + \epsilon)$, $\mu_n = b_n/(b_n - \epsilon)$. One estimates the integrals $I_n(Ac_n, c_n\lambda_n)$, $I_n(c_n\lambda_n, c_n\mu_n)$, and $I_n(c_n\mu_n, 0)$ and finds that they converge to zero as $n \to \infty$. On the other hand, $I_n(0, u) \to I(0, u)$, by Lebesgue, and $(L - \epsilon)(f(0) - \epsilon)\int_{-\infty}^{Ac_n} 1/(1 + x^2)\, dx \leqq I_n(-\infty, Ac_n) \leqq (L + \epsilon)(f(0) + \epsilon)\int_{-\infty}^{Ac_n} 1/(1 + x^2)\, dx$. Hence $I_n(-\infty, u) \to I(-\infty, u)$. This conclusion holds, by a similar argument, if $c_n > 0$.

Case (f). If $L \neq 0$, $\alpha = 2$, the conclusions are the same as in Case (d).

If $L \neq 0$, $\alpha < 2$, $I_n(c_n\lambda_n, c_n\mu_n) \to \infty$, where λ_n and μ_n are defined in the immediately preceding case.

If $L = 0$, since $I_n(-\infty, -\epsilon)$ and $I_n(\epsilon, u)$ converge to zero, by Lebesgue, we have the impermissible situation that $\theta(u)$ is identically zero or is a step-function with a step (of finite or infinite size) at $u = 0$.

Case (g). If $1 \leqq \alpha < 2$, then $I_n(-\infty, u) \to I(-\infty, u)$, by Lebesgue. Suppose $\alpha < 1$. In case $c_n < 0$, let $1/c_n < A < 0$. Then $I_n(-\infty, A) \to 0$ as $A \to -\infty$, uniformly with respect to n, whereas $I_n(A, u) \to I(A, u)$, by Lebesgue. Should c_n be positive, let $1/c_n > A > 0$. Then $I_n(A, \infty) \to 0$ as $A \to \infty$, uniformly with respect to n, whereas $I_n(-\infty, A) \to I(-\infty, A)$. We conclude that if $L \neq 0$ and $\alpha < 1$, $I_n(-\infty, u) \to I(-\infty, u)$.

Case (k). Let $1/c_n < A < B < 0$, if $c_n < 0$ and $0 < A < B < 1/c_n$, if $c_n > 0$. Then, by Lebesgue, $I_n(A, B) \to I(A, B)$.

If $L \neq 0$, $I(A, B) \to \infty$, as $A \to -\infty$ (if $c_n < 0$), or as $B \to \infty$ (if $c_n > 0$). This means that $\theta(u)$ has to be unbounded.

If $L = 0$, $I(A, B) = 0$. But $I_n(\epsilon, \infty) \to I(\epsilon, \infty)$ if c_n is negative, and $I_n(-\infty, -\epsilon) \to I(-\infty, -\epsilon)$ if c_n is positive. Thus $\theta(u)$ is a step-function with a step at zero. This is inadmissible.

Case (m). This case leads to a $\theta(u)$ which is either unbounded or is a step-function with a single step.

It has thus been shown that the only cases that need further investigation are (c) and (g), with $\alpha < 2$, (f) with $\alpha = 2$, (d) and (e). In these cases, we have

$$\theta(u) = \int_{-\infty}^{u} [1/(1 + x^2)][\lim_{n\to\infty} g_n^*(x)]\, dx. \tag{3.7}$$

But, if (3.7) holds in the Cases (d), (e), (f), and (g), $1/(X + c)$ becomes (using (3.4) and [1], p. 168) stable, with one of its characteristic parameters β equal to zero ([1], p. 169). Suppose now, that the characteristic parameter α of $1/X$ is not unity. Let $c \neq 0$. Then the pdf $(1/x^2)g(1/x - c)$ of $1/(X + c)$ is either zero or infinite at $x = 1/c$ (see (iii)). Hence, from the continuity of stable laws, their unimodality ([2]) and the fact that they are not bounded to the right or left unless $|\beta| = 1$ ([3], p. 105 ff., or [5]), we reach the conclusion that the assumption $\alpha \neq 1$, is untenable when the Cases (d), (e), (f), and (g) hold.

Thus if $\alpha \neq 1$, the only case that can hold is (c). Suppose that the situation delineated in (c) occurs. We have, from (3.7) and (c), with $\alpha \neq 2$,

$$\theta(u) = \lambda(c)\int_{-\infty}^{u} [1/(1 + x^2)]|k(c)/x - 1|^{\alpha-1}\, dx, \tag{3.8}$$

where k has been replaced by $k(c)$ to indicate explicitly the dependence on c.

Next, if $\psi_i(t)$, $i = 1, 2, 3$, are the ch.f.'s of the three r.v.'s in (3.2), we obtain the equivalent relation

$$\psi_1(t)\psi_2(t) = \psi_3(t). \tag{3.9}$$

But, by Lemma 1, $\psi_3(t) = \exp(-\mu|A(c)t|^{\alpha})$. Hence, ([1], p. 168)

$$\log \psi_3(t) = i\delta(c)t + \eta(c)\int (e^{itu} - 1 - itu/1 + u^2)|u|^{-\alpha-1}\, du,$$

where $\delta(c)$ and $\eta(c)$ are some functions of c.

On the other hand, by (3.4) and (3.8),

$$\log \psi_1(t) = i\gamma(c)t + \lambda(c)\int (e^{itu} - 1 - itu/1 + u^2)|u|^{-\alpha-1}|k(c) - u|^{\alpha-1}\, du,$$

and $\log \psi_2(t)$ is obtained by replacing c by $-c$ in the right hand side of the last equation.

Hence, from (3.9), using the uniqueness of the Lévy-Khinchin representation ([1], p. 80), we have

$$\eta(c) \equiv \lambda(c)|k(c) - u|^{\alpha-1} + \lambda(-c)|k(-c) - u|^{\alpha-1},$$

an identity which is impossible unless $\alpha = 1$. Thus, the assumption $\alpha \neq 1$ that was made above is invalid, and the proof that Conditions (3.1) and (3.2) are sufficient is complete. That these conditions are necessary is easy to establish. As for the corollary, it follows from the stated assumptions that there exist a and b such that $aX + b$ is symmetric, and the conclusion follows from that of the theorem.

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