A FAMILY OF COMBINATORIAL IDENTITIES

By Lawrence H. Harper

The Rockefeller Institute

Given an ordered n-tuple of real numbers, $(x_1, x_2, \cdots, x_n)$, let σ denote any cyclic permutation of these numbers. If $\sigma = (y_1, \cdots, y_n)$ then σ_j denotes the ordered n-tuple defined by $\sigma_j = (y_j, y_{j-1}, \cdots, y_1, y_n, y_{n-1}, \cdots, y_{j+1})$. In particular, if $\sigma = (y_1, \cdots, y_n)$, then $\sigma_n = (y_n, \cdots, y_1)$. Note that $(\sigma_j)_j = \sigma$, so that the operation is 1-1 and onto.

Now we make the following definition which extends the notation used by R. L. Graham in [3].

Definition 1. $M_{rj}(z_1, \cdots, z_n)$ and $m_{rj}(z_1, \cdots, z_n)$ denote the rth largest and the rth smallest, respectively, among the first j partial sums $z_1, z_1 + z_2, \cdots, z_1 + \cdots + z_j$ for $1 \leqq r \leqq j \leqq n$. Note that $M_{rj} = m_{j-r+1,j}$.

Definition 2. If x is a real number, then

$$\begin{aligned} x^+ &= x \quad \text{if} \quad x \geqq 0 \\ &= 0 \quad \text{if} \quad x < 0 \end{aligned}$$

and

$$\begin{aligned} x^- &= 0 \quad \text{if} \quad x \geqq 0 \\ &= x \quad \text{if} \quad x < 0. \end{aligned}$$

Note that $x = x^+ + x^-$.

Theorem.

$$\sum_\sigma [M_{rj}^+(\sigma) + m_{rj}^-(\sigma_n)] = (j - r + 1)s_n \tag{1}$$

where the sum is taken over all cyclic permutations of $(x_1, \cdots, x_n)$, *a total of* n, *and* $s_n = x_1 + \cdots + x_n$.

This formula, the main result of the note, is a generalization of a combinatorial theorem on partial sums by R. L. Graham [3], which appears here as Corollary 2. Graham had generalized a result of M. Dwass [1] and our extension includes another formula of Dwass' from the same paper, here Corollary 1.

Proof. The proof is based upon two identities:

$$M_{rj}^+(\sigma) + m_{rj}^-(\sigma_j) = M_{r-1,j-1}^+(\sigma) + m_{r-1,j-1}^-(\sigma_j) \tag{2}$$

and

$$M_{1j}^+(\sigma) + m_{1j}^-(\sigma_j) = s_j, \tag{3}$$

where $\sigma = (y_1, \cdots, y_n)$ and $s_j = y_1 + \cdots + y_j$ for $1 \leqq r \leqq j \leqq n$. The intuitive idea behind these identities is a geometrical one, a variant of the reflection prin-

Received 19 April 1965; revised 10 November 1965.

ciple in Feller's book ([2], p. 69). We now proceed to the proof. Let j be fixed, and put

$$A_r = M_{rj}(\sigma) = \max_r (y_1, y_1 + y_2, \cdots, y_1 + \cdots + y_j),$$

$$B_r = m_{rj}(\sigma) = \min_r (y_j, y_j + y_{j-1}, \cdots, y_j + y_{j-1} + \cdots + y_1).$$

Let further, for $1 \leqq k < j$,

$$a_k = \max_k (y_1, y_1 + y_2, \cdots, y_1 + \cdots + y_{j-1}),$$

$$b_k = \min_k (y_j + y_{j-1} + \cdots + y_2, \cdots, y_j + y_{j-1}, y_j).$$

Clearly, $a_k + b_k = s$, where $s = y_1 + y_2 \cdots + y_j$.

Since $a_{r-1} = M_{r-1,j-1}(\sigma)$, $b_{r-1} = m_{r-1,j-1}(\sigma_j)$, we must prove $A_r^+ + B_r^- = a_{r-1}^+ + b_{r-1}^-$. As is easily seen

$$A_r = \max_2 (a_{r-1}, a_r, s), \qquad B_r = \min_2 (b_{r-1}, b_r, s);$$

(if $r = j$ we can take b_j as some very large number, $a_j = s - b_j$ very small). For shortness, put $h = a_{r-1} - a_r$, $a = a_{r-1}$, $b = b_{r-1}$. Then $h \geqq 0$, while $a + b = s$, $a_r = a - h$, $b_r = b + h$. Thus, we must prove that

$$(*) \quad [\max_2 (a, a - h, a + b)]^+ + [\min_2 (b, b + h, a + b)]^- = a^+ + b^-,$$

whenever $h \geqq 0$. If either $a < 0$ or $b > 0$ then

$$[\max_2 (a, a - h, a + b)]^+ = a^+, \qquad [\min_2 (b, b + h, a + b)]^- = b^-.$$

Thus, in proving $(*)$, we may assume that $a \geqq 0$, $b \leqq 0$, in which case the right hand side of $(*)$ equals $a + b$. Moreover,

$$[\max_2 (a, a - h, a + b)]^+ = [\max (a + b, a - h)]^+ = \max (a + b, a - h, 0).$$

Similarly,

$$[\min_2 (b, b + h, a + b)]^- = [\min (b + h, a + b)]^- = \min (0, b + h, a + b).$$

$$= a + b - \max (a + b, a - h, 0).$$

This completes the proof of $(*)$.

From (2) and (3) we conclude the theorem in the following way:

$$\textstyle\sum_\sigma [M_{rj}^+(\sigma) + m_{rj}^-(\sigma_n)] = \sum_\sigma [M_{rj}^+(\sigma) + m_{rj}^-(\sigma_j)]$$

since all of the mappings $\sigma \to \sigma_j$ are 1-1,

$$= \textstyle\sum_\sigma [M_{r-1,j-1}^+(\sigma) + m_{r-1,j-1}^-(\sigma_j)] \qquad \text{by (2)}$$

$$= \textstyle\sum_\sigma [M_{r-1,j-1}^+(\sigma) + m_{r-1,j-1}^-(\sigma_n)].$$

So

$$\textstyle\sum_\sigma [M_{rj}^+(\sigma) + m_{rj}^-(\sigma_n)] = \sum_\sigma [M_{1,j-r+1}^+(\sigma) + m_{1,j-r+1}^-(\sigma_n)] \qquad \text{by induction}$$

$$= \textstyle\sum_\sigma s_{j-r+1} \qquad \text{by (3)}$$

$$= (j - r + 1)s_n.$$

The last equality being valid since each x_i will appear in exactly $(j - r + 1)$ of the s_{j-r+1}.

COROLLARY 1.

$$\sum_\sigma M^+_{nn}(\sigma) = s_n^+, \qquad (4)$$

$$\sum m^-_{nn}(\sigma) = s_n^-.$$

PROOF. Letting $r = j = n$ in (1) we have $\sum_\sigma [M^+_{nn}(\sigma) + m^-_{nn}(\sigma_n)] = s_n$. But assume $s_n \geqq 0$, then $m^-_{nn}(\sigma) = 0$, and symmetrically, if $s_n < 0$, $M^+_{nn}(\sigma) = 0$. This is exactly Theorem 1 of Dwass [1].

COROLLARY 2.

$$\sum [M^+_{rn}(\sigma) - M^+_{r,n-1}(\sigma)] = s_n^+, \qquad 1 < r < n. \qquad (5)$$

PROOF. Let $j = n, n - 1$ in (1) and take the difference

$$\sum_\sigma [M^+_{rn}(\sigma) - M^+_{r,n-1}(\sigma)] - \sum_\sigma [m^-_{rn}(\sigma_n) - m^-_{r,n-1}(\sigma_n)] = s_n.$$

If $s_n \geqq 0$ $m^-_{rn}(\sigma_n) = m^-_{r,n-1}(\sigma_n)$, so $\sum_\sigma [m^-_{rn}(\sigma_n) - m^-_{r,n-1}(\sigma_n)] = 0$ and so $\sum_\sigma [M^+_{rn}(\sigma) - M^+_{r,n-1}(\sigma)] = s_n$. If $s_n < 0$, by the same argument $\sum_\sigma [M^+_{rn}(\sigma) - M^+_{r,n-1}(\sigma)] = 0$. This is Theorem 1 of Graham [3].

NOTE. Corollary 1 is contained in Corollary 2 if M_{rn} is defined to be zero for $r > n$. However, by applying Corollary 1 one obtains Corollary 2 for $1 \leqq r \leqq n$.

COROLLARY 3.

$$\sum_\sigma [M_{rj}(\sigma) + m_{rj}(\sigma_n)] = (j + 1)s_n \qquad (6)$$

$$\sum_\sigma \{|M_{rj}(\sigma)| - |m_{rj}(\sigma_n)|\} = (j - 2r + 1)s_n \quad \textit{where } 1 \leqq r \leqq j \leqq n.$$

PROOF. Apply (1) with r replaced by $j - r + 1$ and σ by σ_n, and note that $M_{j-r+1,j}(\sigma_n) = m_{r,j}(\sigma_n)$, $m_{j-r+1,j}(\sigma) = M_{r,j}(\sigma)$, then

$$\sum_\sigma [m^+_{rj}(\sigma_n) + M^-_{rj}(\sigma)] = rs_n.$$

Adding and subtracting this from (1) gives the results advertised. Following Kimme [4] each of these combinatorial identities corresponds to a distribution-free identity for the partial sums of certain kinds of r. v.'s:

$$E(M^+_{rj} + m^-_{rj} \mid s_n) = [(j - r + 1)/n]s_n \qquad (1')$$

$$E(M^+_{nn} \mid s_n) = s_n^+/n \qquad (4')$$

$$E(m^-_{nn} \mid s_n) = s_n^-/n$$

$$E(M^+_{rn} - M^+_{r,n-1} \mid s_n) = s_n^+/n \qquad (5')$$

$$E(M_{rj} + m_{rj} \mid s_n) = [(j + 1)/n]s_n \qquad (6')$$

$$E(|M_{rj}| - |m_{rj}| \mid s_n) = [(j - 2r + 1)/n]s_n.$$

(1′) and the two formulas (6′) hold whenever $x_1, \cdots, x_n$ are random variables whose distribution is invariant under cyclic permutations and also under a reflection which sends x_k into x_{n+1-k}. The two formulas (4′) and also (5′) hold as soon as the above joint distribution is invariant under cyclic permutations.

CONCLUSION. There are interesting applications of some of these formulas in the literature [3]. With the wealth of new identities here more applications should follow.

REFERENCES

[1] DWASS, M. (1962). A fluctuation theorem for cyclic random variables. *Ann. Math. Statist.* **33** 1450–1454.

[2] FELLER, W. (1963). *An Introduction to Probability Theory and Its Applications*, (2nd ed.). Wiley, New York.

[3] GRAHAM, R. L. (1963). A combinatorial theorem for partial sums. *Ann. Math. Statist.* **34** 1600–1602.

[4] KIMME, E. G. (1963). Relations between combinatorial and distribution-free identities. Bell Telephone Laboratories. Private communication.