ON A MINIMAL ESSENTIALLY COMPLETE CLASS OF EXPERIMENTS[1]

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1. Introduction and summary. The purpose of this paper is to show that a certain class of experiments is minimal essentially complete asymptotically and to demonstrate that this is not generally the case for finite sample sizes.

The model to be discussed involves *linear experiments* consisting of uncorrelated observations. That is, the experimenter may choose n uncorrelated random variables $Y(x_1), Y(x_2), \cdots, Y(x_n)$ with expectation,

$$E[Y(x)] = x'\theta \tag{1.1}$$

where θ is a parameter vector in k dimensional space $E^{(k)}$ and x is a vector also in $E^{(k)}$, to some extent to be chosen by the experimenter, and the variance $V[Y(x)] = \sigma^2$. An experiment, e, of size n is fully specified, using the terminology of Elfving [6], by a spectrum $(x_1, x_2, \cdots, x_r)$ consisting of the different x's and an allocation $(n_1, n_2, \cdots, n_r)$ where $n_1 + n_2 + \cdots + n_r = n$. Thus, e can be represented by $e = (n_1, n_2, \cdots, n_r; x_1, x_2, \cdots, x_r)$. We are concerned with the case where x is restricted to lie in a set A. The questions to be discussed relate to the choice of the x's in A. Some results in this direction have previously been obtained and are discussed by Elfving [5], [6], [7], Ehrenfeld [2], [3] and Kiefer [9], [10].

The information matrix, $F(e)$, associated with experiment e, is given by

$$F(e) = n_1x_1x_1' + \cdots + n_rx_rx_r' = n(p_1x_1x_1' + \cdots + p_rx_rx_r') \tag{1.2}$$

where $p_j = n_j/n$ is called the relative allocation at x_j. Also, $p_j \geqq 0$ and $p_1 + p_2 + \cdots + p_r = 1$. Strictly speaking, in the *exact* theory, the p_j's can only range over multiples of $1/n$. However, in the *approximate* or *asymptotic* theory, the p_j's have been allowed to range continuously from 0 to 1, see Elfving [5], [6] and Kiefer [9], [10]. We will show later that the exact and asymptotic theory differ in essential ways.

We will assume throughout that the set A is symmetric about the origin. This can be done, without essential restriction, since $-Y(x)$ is automatically available with $Y(x)$, see Elfving [6].

Before stating some results we have to introduce some notation for certain classes of experiments. In the approximate theory we denote by $\mathcal{E}[A]$ the set of experiments e with the x's restricted to be in A. We assume that any $e \in \mathcal{E}[A]$ is described by a spectrum $(x_1, x_2, \cdots, x_r)$ consisting of a finite number of x's in A and a relative allocation $(p_1, p_2, \cdots, p_r)$. The sample size plays no role

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here since we are dealing with the asymptotic case. Also, the value of r is part of the choice of the experiment.

In the exact theory, we denote by $\mathcal{E}_N[A]$ the set of experiments e where the x's are restricted to be in set A and the sample size $n \leqq N$.

To compare experiments, we introduce a partial ordering $e_1 \geqq e_2$ which will mean that

$$V_{e_1}[t'\hat{\theta}] \leqq V_{e_2}[t'\hat{\theta}] \quad \text{for all } t \tag{1.3}$$

where $V_e[t'\hat{\theta}]$ denotes the variance of the least square estimate of $t'\theta$ with experiment e. When $t'\theta$ is not estimable with respect to e, $V_e[t'\hat{\theta}]$ is set equal to infinity. It was shown, Ehrenfeld [3], that relation (1.3) is equivalent to $F(e_1) - F(e_2)$ being a non-negative definite matrix.

Finally, we consider comparing two classes of experiments $\mathcal{E}_1$ and $\mathcal{E}_2$. We say that $\mathcal{E}_1$ is *essentially complete* with respect to $\mathcal{E}_2$ when for any $e_2 \varepsilon \mathcal{E}_2$ there exists an $e_1 \varepsilon \mathcal{E}_1$ such that $e_1 \geqq e_2$. Furthermore, $\mathcal{E}_1$ is said to be a *minimal essentially complete* class when no proper subset of $\mathcal{E}_1$ is complete with respect to $\mathcal{E}_2$.

We also define a weaker kind of minimality, designated by minimal essential completeness (W), in the exact or asymptotic case. Suppose $\mathcal{E}$, depends on a set B of experimental points. The essentially complete class $\mathcal{E}_1$, is *minimal essentially complete* (W) with respect to $\mathcal{E}_2$ if no subset of $\mathcal{E}_1$, depending on a proper subset of B, is essentially complete with respect to $\mathcal{E}_2$. That is, one cannot remove any elements of B without losing essential completeness.

Various results concerning essentially complete classes were proven and discussed by Ehrenfeld [3], Elfving [6] and Kiefer [9], [10].

Let $E(A)$ denote the extreme points of the set A. A point x is an extreme point of A if it is not a convex combination of two or more points of the set A. We will show when A is compact that, in the asymptotic case, $\mathcal{E}[E(A)]$ is minimal essentially complete (W) with respect to $\mathcal{E}[A]$. Furthermore, we will demonstrate, with an example and some general theorems, that in the exact case $\mathcal{E}_N[E(A)]$ is not necessarily even essentially complete with respect to $\mathcal{E}_N[A]$.

2. The asymptotic case. Before proceeding to the main result, we state as Theorem 1 a result which follows essentially from Theorem 2.3 of Ehrenfeld [3]. Denote by $P(b_1, b_2, \cdots, b_m)$ the polyhedron generated by the vectors $b_1, b_2, \cdots, b_m$. That is, $P(b_1, b_2, \cdots, b_m)$ is the convex hull of $b_1, b_2, \cdots, b_m$.

THEOREM 1. *In the asymptotic case,* $\mathcal{E}[b_1, b_2, \cdots, b_m]$ *is essentially complete with respect to* $\mathcal{E}[P(b_1, b_2, \cdots, b_m)]$.

This result means essentially, in the asymptotic case, that given any $e = (p_1, \cdots, p_r; x_1, x_2, \cdots, x_r)$ with $x_1, x_2, \cdots, x_r$ vectors in the polyhedron generated by some vectors $b_1, b_2, \cdots, b_m$, then there exists a relative allocation $(p_1^*, p_2^*, \cdots, p_m^*)$ such that,

$$e^* = (p_1^*, p_2^*, \cdots, p_m^*; b_1, b_2, \cdots, b_m) \geqq e$$
$$= (p_1, p_2, \cdots, p_r; x_1, x_2, \cdots, x_r).$$

For a proof of this result let

$$x_i = \sum_{j=1}^{m} \lambda_{ij} b_j \quad \text{with} \quad \lambda_{ij} \geqq 0 \quad \text{and} \quad \sum_j \lambda_{ij} = 1.$$

Then,

$$t'(x_i x_i')t = (t'x_i)^2 = [\sum_{j=1}^{m} \lambda_{ij}(t'b_j)]^2 \leqq \sum_{j=1}^{m} \lambda_{ij}(t'b_j)^2.$$

Hence,

$$t'F(e)t \leqq n \sum_{i=1}^{r} p_i(\sum_{j=1}^{m} \lambda_{ij}(t'b_j)^2) = n \sum_{j=1}^{m} p_j^*(t'b_j)^2 = t'F(e^*)t,$$

where $p_j^* = \sum_{i=1}^{r} p_i \lambda_{ij}$ and $p_j^* \geqq 0$ with $\sum_{j=1}^{m} p_j^* = 1$.

We can now state the main result for the asymptotic case.

THEOREM 2. *In the asymptotic case, if A is a compact set in $E^{(k)}$, (symmetric about the origin) then $\mathcal{E}[E(A)]$ is minimal essentially complete (W) with respect to* $\mathcal{E}$[A].

PROOF. It is sufficient to demonstrate the result when A is compact and convex since it can be shown, using results from the theory of convex sets, that if $C(A)$ denotes the convex hull of A (i.e., the smallest convex set containing A) then $E(C(A)) = E(A)$.

We first show the essential completeness of $\mathcal{E}[E(A)]$ with respect to $\mathcal{E}[A]$. Let $x_1, x_2, \cdots, x_r$ be vectors in A. Again, from the theory of convex sets, it may be shown that the compact convex set A is spanned by its extreme points. That is, $A = C(E(A))$, see Eggleston [1] and Karlin [8]. Furthermore, from Caratheodory's theorem, see Eggleston [1], it follows that each x_j is included in a polyhedron generated by no more than $k + 1$ extreme points of A. Hence, all the vectors $x_1, x_2, \cdots, x_r$ are included in a polyhedron generated by no more than $r(k + 1)$ elements of $E(A)$. The essential completeness of $\mathcal{E}[E(A)]$ with respect to $\mathcal{E}[A]$ now follows basically from Theorem 1.

We now demonstrate that $\mathcal{E}[E(A)]$ is minimal essentially complete. Let $v \in E(A)$ and $E'(A) = \{x \mid x \in E(A) \text{ and } x \neq v \text{ or } -v\}$. We must show that $\mathcal{E}[E'(A)]$ is not essentially complete. To do this it is sufficient to produce an experiment e^* not in $\mathcal{E}[E'(A)]$ such that for each $e \in \mathcal{E}[E'(A)]$ there is a t_0 with $t_0'F(e)t_0 < t_0'F(e^*)t_0$, (since then $F(e) - F(e^*)$ cannot be non-negative definite). Note that t_0 may depend on e.

Consider $e^* = (1; v)$ which is not in $\mathcal{E}[E'(A)]$ and any experiment $e = (p_1, p_2, \cdots, p_r; x_1, x_2, \cdots, x_r)$ with $x_j \in E'(A)$, (i.e., $e \in \mathcal{E}[E'(A)]$). Now,

$$t'F(e^*)t = nt'(vv')t = n(t'v)^2 \quad \text{and} \tag{2.1}$$

$$t'F(e)t = nt'(p_1x_1x_1' + \cdots + p_rx_rx_r')t = n\sum_{j=1}^{r} p_j(t'x_j)^2.$$

Using the theorem on separating hyperplanes in the theory of convex sets, see Eggleston [1] and Karlin [8], and the symmetry around the origin, it follows that there exists a t_0 such that,

$$(t_0'x_j)^2 < (t_0'v)^2 \qquad (j = 1, 2, \cdots, r). \tag{2.2}$$

Hence,

$$(2.3)\qquad t_0'F(e)t_0 = n\sum_{j=1}^{r} p_j(t_0'x_j)^2 < n(t_0'v)^2 = t_0'F(e^*)t_0,$$

giving the required result.

Theorem 3.6 in [9] shows that the minimality result is not generally true in the stronger sense.

3. The finite sample case. Before stating some general results concerning the non-essential completeness of $\mathcal{E}_N[b_1, b_2, \cdots, b_m]$ with respect to $\mathcal{E}_N[P(b_1, b_2, \cdots, b_m)]$ we consider some examples.

EXAMPLE 1. Let $b_1' = (1, 1)$; $b_2' = (1, -1)$; $b_3' = -b_1' = (-1, -1)$ and $b_4' = -b_2 = (-1, 1)$. Here $k = 2$ and $m = 4$. We now study the special case where $N \equiv 1 \bmod 4$. That is $N = 4n + 1$. Consider the particular experiment $e^* = (n, n, n, n, 1; b_1, b_2, b_3, b_4, v)$ where $v' = (1, 0)$. The vector $v \varepsilon P(b_1, b_2, b_3, b_4)$ and e^* is not in $\mathcal{E}_N[b_1, b_2, b_3, b_4]$. It is clear that, for any $e = (n_1, n_2, n_3, n_4; b_1, b_2, b_3, b_4)$, we have $\min(n_1 + n_3; n_2 + n_4) \leqq 2n$. Suppose that $\min(n_1 + n_3; n_2 + n_4) = n_1 + n_3$. Then, if we choose any $t = t_0$ orthogonal to vector b_2 we have,

$$(3.1)\qquad t_0'F(e)t_0 \leqq 2n(t_0'b_1)^2 \quad\text{and}\quad t_0'F(e^*)t_0 = 2n(t_0'b_1)^2 + (t_0'v)^2.$$

Hence, since $(t_0'v)^2 > 0$ we have $t_0'F(e)t_0 < t_0'F(e^*)t_0$. If $\min(n_1 + n_3; n_2 + n_4) = n_2 + n_4$, a similar procedure works with any t orthogonal to b_1. In any case, no choice of $e \varepsilon \mathcal{E}_N[b_1, b_2, b_3, b_4]$ will make $F(e) - F(e^*)$ a non-negative definite matrix.

This example illustrates the fact there can be an infinite number of values of N for which $\mathcal{E}_N[b_1, b_2, \cdots, b_m]$ is not essentially complete with respect to $\mathcal{E}_N[P(b_1, b_2, \cdots, b_m)]$.

EXAMPLE 2. Let $b_1 = (1, 1)$ and $b_2 = -b_1 = (-1, -1)$. Here, $k = 2$ and $m = 2$. In this case, it is clear that for all N we have $\mathcal{E}_N[b_1, b_2]$ essentially complete with respect to $\mathcal{E}_N[P(b_1, b_2)]$. This is also a special case of Theorem 2.1 of Ehrenfeld [3].

We now prove the non-essential completeness of $\mathcal{E}_N[b_1, b_2, \cdots, b_m]$ with respect to $\mathcal{E}_N[P(b_1, b_2, \cdots b_m)]$ in some general contexts. However, the complete story remains to be told. In the following theorem let $d(b_1, \cdots, b_m)$ denote the dimension of the vector space spanned by $b_1, b_2, \cdots, b_m$.

THEOREM 3. *If* $N < d(b_1, b_2, \cdots, b_m)$ *then* $\mathcal{E}_N[b_1, b_2, \cdots, b_m]$ *is not essentially complete with respect to* $\mathcal{E}_N[P(b_1, b_2, \cdots, b_n)]$.

PROOF. Any experiment $e \varepsilon \mathcal{E}_N[b_1, b_2, \cdots, b_m]$ must include a set of no more than N of the b's. Denote these by $b_{i_1}, b_{i_2}, \cdots, b_{i_s}$. Let the allocation, associated with these vectors, be $n_1, n_2, \cdots, n_s$ with $n_1 + n_2 + \cdots + n_s \leqq N$. For any such set of b's consider the vector space, V, orthogonal to the space spanned by these b's. Since $N < d(b_1, \cdots, b_m)$ the space V is not empty. There are a finite number of such subsets of b's and hence a finite number of V's, say $V_1, V_2, \cdots, V_r$.

Choose a vector $v \neq 0$ in $P(b_1, b_2, \cdots, b_m)$, not equal to any of the b's and also not in any one of the subspaces $V_1, V_2, \cdots, V_r$.

Now, consider experiment $e^* = (N; v)$ which is not in $\mathcal{E}_N[b_1, b_2, \cdots, b_m]$. With any $e \varepsilon \mathcal{E}_N[b_1, b_2, \cdots, b_m]$ we have:

$$(3.2) \qquad t'F(e)t = n_1(t'b_{i_1})^2 + n_2(t'b_{i_2})^2 + \cdots + n_s(t'b_{i_s})^2,$$
$$t'F(e^*)t = N(t'v)^2.$$

For any such e we can choose $t = t_0$ as a vector in the space orthogonal to the space spanned by the set $b_{i_1}, b_{i_2}, \cdots, b_{i_s}$ (i.e., one of the V's). Hence,

$$(3.3) \quad (t_0'b_{i_1})^2 = (t_0'b_{i_2})^2 = \cdots = (t_0'b_{i_s}) = 0, \quad \text{and} \quad (t_0'v)^2 > 0$$

because of the way v was chosen. Thus,

$$(3.4) \qquad t_0'F(e)t_0 - t_0'F(e^*)t_0 = -N(t_0'v)^2 < 0,$$

which proves the result.

Before stating the next theorem we note that, because of the symmetry of the set $P(b_1, b_2, \cdots, b_m)$, we can take $m = 2w$. That is, the b's come in pairs. We will order the b's so that $b_{w+j} = -b_j$ for $j = 1, 2, \cdots, w$.

THEOREM 4. *If $b_1, b_2, \cdots, b_w$ are independent vectors and $N \equiv s \bmod m$ where $0 < s < w$, then $\mathcal{E}_N[b_1, b_2, \cdots, b_m]$ is not essentially complete with respect to $\mathcal{E}_N[P(b_1, b_2, \cdots, b_m)]$.*

PROOF. Let $N = mn + s$ with $0 < s < w$. Furthermore, let V_i denote the vector space spanned by the vectors $b_1, b_2, \cdots, b_w$ excluding b_i for $i = 1, 2, \cdots, w$. We choose $e^* \varepsilon \mathcal{E}_N[P(b_1, \cdots, b_m)]$ and not in $\mathcal{E}_N[b_1, b_2, \cdots, b_m]$ as $e^* = (n, n, \cdots, n, v; b_1, b_2, \cdots, b_m, s)$ where $v \neq 0$, in $P(b_1, b_2, \cdots, b_m)$, not orthogonal to any of $V_1, V_2, \cdots, V_w$, and not equal to any of the b's.

For any experiment $e = (n_1, n_2, \cdots, n_m; b_1, b_2, \cdots, b_m) \varepsilon \mathcal{E}_N[b_1, b_2, \cdots, b_m]$ we have $n_1 + n_2 + \cdots + n_m \leqq N$. Also, it is clear that min $(n_1 + n_{w+1}; n_2 + n_{w+2}; \cdots; n_w + n_m) \leqq 2n$. That is, there is an i for which $n_i + n_{w+i} \leqq 2n$. For any such experiment e we can choose $t = t_0$ as a vector orthogonal to V_i. Then,

$$t_0'F(e)t_0 \leqq 2n(t_0'b_i)^2 \quad \text{and} \quad t_0'F(e^*)t_0 = 2n(t_0'b_i)^2 + s(t_0'v)^2.$$

The result follows since $s(t_0'v)^2 > 0$, from the way t_0 and v were chosen.

One of the implications of Theorems 3 and 4 is that examples can be constructed where $\mathcal{E}_N[b_1, b_2, \cdots, b_m]$ is not essentially complete with respect to $\mathcal{E}_N[P(b_1, b_2, \cdots, b_m)]$ for an infinite number of N. From this it also follows, in particular for these examples, that $\mathcal{E}_N[E(A)]$ is not essentially complete with respect to $\mathcal{E}_N[A]$. For these examples A is, of course, $P(b_1, b_2, \cdots, b_m)$.

Some open questions which remain are to fully characterize essentially complete classes of experiments for the exact, finite sample, case. Some theorems relating to essential completeness, in the exact case, are given by Ehrenfeld [3]. However, further questions remain, particularly concerning minimal essentially complete classes of experiments.

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