LIMITING DISTRIBUTIONS FOR SOME RANDOM WALKS ARISING IN LEARNING MODELS

By M. Frank Norman[1]

Institute for Mathematical Studies in the Social Sciences

1. Introduction and summary. Associated with certain of the learning models introduced by Bush and Mosteller [1] are random walks $p_1, p_2, p_3, \cdots$ on the closed unit interval with transition probabilities of the form

$$P[p_{n+1} = p_n + \theta_1(1 - p_n) \mid p_n] = \varphi(p_n) \tag{1}$$

and

$$P[p_{n+1} = p_n - \theta_2 p_n \mid p_n] = 1 - \varphi(p_n) \tag{2}$$

where $0 < \theta_1, \theta_2 < 1$ and φ is a mapping of the closed unit interval into itself. In the experiments to which these models are applied, response alternatives A_1 and A_2 are available to a subject on each of a sequence of trials, and p_n is the probability that the subject will make response A_1 on trial n. Depending on which response is actually made, one of two events E_1 or E_2 ensues. These events are associated, respectively, with the increment $p_n \to p_n + \theta_1(1 - p_n)$ and the decrement $p_n \to p_n - \theta_2 p_n$ in A_1 response probability. The conditional probabilities π_{ij} of event E_j given response A_i do not depend on the trial number n. Thus (1) and (2) are obtained with $\varphi(p) = \pi_{11}p + \pi_{21}(1 - p)$.

Since the linearity of the functions φ which arise in this way is of no consequence for the work presented in this paper, we will assume instead simply that

$$\varphi \,\varepsilon\, C^2([0, 1]). \tag{3}$$

We impose one further restriction on φ which excludes some cases of interest in learning theory:

$$\epsilon_1 = \min_{0 \leq p \leq 1} \varphi(p) > 0 \quad \text{and} \quad \epsilon_2 = \max_{0 \leq p \leq 1} \varphi(p) < 1. \tag{4}$$

It follows from a theorem of Karlin ([5], Theorem 37) that under (1)–(4) the distribution function $F^{(n)}_{\theta_1,\theta_2,\varphi}$ of p_n (which depends, of course, on the distribution F of p_1) converges as n approaches infinity to a distribution $F_{\theta_1,\theta_2,\varphi}$ which does not depend on F. It is with the distributions $F_{\theta_1,\theta_2,\varphi}$ that the present paper is concerned.

Very little is known about distributions of this family, though some results may be found in Karlin [5], Bush and Mosteller [1], Kemeny and Snell [6], Estes and Suppes [3], and McGregor and Hui [8]. The only theorem in the literature directly relevant to the present work is one of McGregor and Zidek [9] as a consequence of which, in the case $\theta_1 = \theta_2 = \theta$, $\varphi(p) \equiv \frac{1}{2}$,

Received 14 July 1965.

[1] Now at the University of Pennsylvania.

$$\lim_{\theta\to 0} \lim_{n\to\infty} P[\theta^{-\frac{1}{2}}(p_n - \tfrac{1}{2}) \leqq x] = \Phi(8^{\frac{1}{2}}x)$$

where Φ denotes the standard normal distribution function; that is, the distribution $F_{\theta,\theta,\frac{1}{2}}(\theta^{\frac{1}{2}}x + \frac{1}{2})$ converges to a normal distribution as the "learning rate" parameter θ tends to 0. We will prove, by means of another method, that this phenomenon is of much greater generality. Theorem 1 below shows that, for any positive constant ζ and any φ with

$$\max_{0\leqq p\leqq 1} \varphi'(p) < \min(1, \zeta)/\max(1, \zeta)$$

there is a constant ρ such that $F_{\theta,\zeta\theta,\varphi}(\theta^{\frac{1}{2}}x + \rho)$ converges to a normal distribution as $\theta \to 0$. A nonnormal limit is obtained if θ_1 approaches 0 while θ_2 remains fixed as is shown in Theorem 2. In this case $F_{\theta_1,\theta_2,\varphi}(\theta_1 x)$ converges to an infinite convolution of geometric distributions.

If $f(p, \theta) = p + \theta(1 - p)$ then (1) and (2) can be written in the form

$$P[p_{n+1} = f(p_n, \theta_1) \mid p_n] = \varphi(p_n) \tag{5}$$

and

$$P[p_{n+1} = 1 - f(1 - p_n, \theta_2) \mid p_n] = 1 - \varphi(p_n). \tag{6}$$

In Section 4 it is shown that the linearity of $f(p, \theta)$ in p and θ is not essential to the phenomena discussed above. Theorems 3 and 4 present generalizations of Theorems 2 and 1, respectively, to "learning functions" $f(p, \theta)$ subject only to certain fairly weak axioms.

A somewhat different learning model, Estes [2] N-element pattern model, leads to a finite Markov chain $p_1, p_2, p_3, \cdots$ with state space $S_N = \{jN^{-1}: j = 0, 1, \cdots, N\}$ and transition probabilities

$$P[p_{n+1} = p_n + N^{-1} \mid p_n] = \varphi(p_n), \tag{7}$$

$$P[p_{n+1} = p_n - N^{-1} \mid p_n] = \psi(p_n), \tag{8}$$

and

$$P[p_{n+1} = p_n \mid p_n] = 1 - \varphi(p_n) - \psi(p_n) \tag{9}$$

where $\varphi(p) = c\pi_{21}(1 - p)$, $\psi(p) = c\pi_{12}p$, $0 < c \leqq 1$, and for the sake of this discussion we suppose that $0 < \pi_{12}, \pi_{21} < 1$. In this case a limiting distribution $F_{N,\varphi,\psi}$ of p_n as $n \to \infty$ exists and is independent of the distribution of p_1 by a standard theorem on Markov chains. Estes [2] showed that the limit is binomial over S_N with mean $r = \pi_{21}/(\pi_{12} + \pi_{21})$. It then follows from the central limit theorem that

$$\lim_{N\to\infty} \lim_{n\to\infty} P[N^{\frac{1}{2}}(p_n - r) \leqq x] = \Phi[x/(r(1 - r))^{\frac{1}{2}}].$$

In Section 5 it is shown that our method permits an extension of this result to much more general φ and ψ.

2. Lemma 1 and Theorem 1. The following lemma is requisite to the proof of Theorem 1.

LEMMA 1. *Suppose that* $\{p_n\}$ *satisfies* (1)–(4). *Suppose, in addition, that* $\zeta > 0$ *and that*

$$\varphi'(p) < [\varphi(p) + (1 - \varphi(p))\zeta]/[(1 - p) + p\zeta] \tag{10}$$

for all $0 \leqq p \leqq 1$. *Then for* $\theta > 0$ *the equation*

$$E[p_{n+1} - p_n \mid p_n = \rho] = 0 \tag{11}$$

has a unique root $\rho = \rho_{\zeta,\varphi}$ *in* $(0, 1)$ *and*

$$\int_{-\infty}^{\infty} (p - \rho)^2 \, dF_{\theta,\zeta\theta,\varphi}(p) = O(\theta).$$

PROOF. Since ζ and φ are fixed throughout the proof we may write

$$E[p_{n+1} - p_n \mid p_n = p] = V(p, \theta) \tag{12}$$

and $F_{\theta,\zeta\theta,\varphi} = F_\theta$. We have $V(p, \theta) = \theta W(p)$ where

$$W(p) = (1 - p)\varphi(p) - \zeta p(1 - \varphi(p)). \tag{13}$$

Now $W(0) = \varphi(0) > 0$, $W(1) = -\zeta(1 - \varphi(1)) < 0$, and by (10)

$$W'(p) = \varphi'(p)[(1 - p) + \zeta p] - [\varphi(p) + \zeta(1 - \varphi(p))] < 0 \tag{14}$$

for all $0 \leqq p \leqq 1$. Thus there is a unique $\rho = \rho_{\zeta,\varphi}$ in $(0, 1)$ such that $W(\rho) = 0$. This constant is obviously the unique root of (11).

To prove the second part of the lemma, we first develop a recursion relation for $E[(p_n - \rho)^2]$. Thus

$$\begin{aligned} E[(p_{n+1} - \rho)^2] \\ &= E[(p_n - \rho)^2] + 2E[(p_n - \rho)(p_{n+1} - p_n)] + E[(p_{n+1} - p_n)^2] \\ &= E[(p_n - \rho)^2] + 2E[(p_n - \rho)V(p_n, \theta)] + E[(p_{n+1} - p_n)^2]. \end{aligned} \tag{15}$$

Now $E[(p_{n+1} - p_n)^2] \leqq \max(1, \zeta^2)\theta^2$, thus letting $n \to \infty$ in (15) we obtain

$$0 = 2 \int_{-\infty}^{\infty} (p - \rho)V(p, \theta) \, dF_\theta(p) + O(\theta^2). \tag{16}$$

Expanding $V(p, \theta)$ around $p = \rho$ we obtain

$$2 \int_{-\infty}^{\infty} (p - \rho)^2(-(\partial/\partial p)V(p^*, \theta)) \, dF_\theta(p) = O(\theta^2)$$

where p^* is between p and ρ so that

$$2 \min_{0 \leqq p \leqq 1} (-(\partial/\partial p)V(p, \theta)) \int_{-\infty}^{\infty} (p - \rho)^2 \, dF_\theta(p) = O(\theta^2)$$

or

$$2(\min_{0 \leqq p \leqq 1} - W'(p)) \int_{-\infty}^{\infty} (p - \rho)^2 \, dF_\theta(p) = O(\theta).$$

Since by (14) and (4) $\min_{0 \leqq p \leqq 1} - W'(p) > 0$, the lemma is established.

This lemma is of some practical importance in its own right. To obtain an approximation $\hat{p}$ to $\lim_{n \to \infty} E[p_n]$ learning theorists are often driven to treat $E[p_{n+1} \mid p_n]$ as though it were linear in p_n thus permitting the replacement of the equation

$$\lim_{n\to\infty} E[p_{n+1}] = \lim_{n\to\infty} E[E[p_{n+1} \mid p_n]]$$

by

$$\hat{p} = E[p_{n+1} \mid p_n = \hat{p}].$$

The estimate $\hat{p}$ thus obtained is termed an *expected operator approximation* (Bush and Mosteller [1]). Since $\hat{p}$ is precisely the ρ of Lemma 1, we see that the lemma provides a justification for expected operator approximation when the learning rates θ_1 and θ_2 are small.

We are now ready to prove

THEOREM 1. *Under the hypotheses of Lemma* 1

$$\lim_{\theta\to 0} F_{\theta,\zeta\theta,\varphi}(\theta^{\frac{1}{2}}x + \rho) = \Phi(x/\sigma)$$

where $\sigma^2 = N(\rho)/2\,|W'(\rho)|$, $W'(\rho)$ *is given by* (14), *and* $N(p)$ *by* (23) *below.*

PROOF. We begin by writing a recursion for the characteristic function of $\theta^{-\frac{1}{2}}(p_n - \rho)$:

$$(17)\quad E[\exp(i\theta^{-\frac{1}{2}}(p_{n+1} - \rho)t)] = E[\exp(i\theta^{-\frac{1}{2}}(p_n - \rho)t)E[\exp(i\theta^{-\frac{1}{2}}(p_{n+1} - p_n)t) \mid p_n]].$$

Defining $Y(p, \theta, t)$ by

$$(18)\quad Y(p, \theta, t) = E[\exp(i\theta^{-\frac{1}{2}}(p_{n+1} - p_n)t) \mid p_n = p]$$

and letting $n \to \infty$ we obtain

$$\int_{-\infty}^{\infty} \exp(i\theta^{-\frac{1}{2}}(p - \rho)t)\, dF_\theta(p) = \int_{-\infty}^{\infty} \exp(i\theta^{-\frac{1}{2}}(p - \rho)t)Y(p, \theta, t)\, dF_\theta(p)$$

or

$$(19)\quad \int_{-\infty}^{\infty} e^{ixt}\, dG_\theta(x) = \int_{-\infty}^{\infty} e^{ixt} Y(\theta^{\frac{1}{2}}x + \rho, \theta, t)\, dG_\theta(x)$$

where

$$(20)\quad G_\theta(x) = F_\theta(\theta^{\frac{1}{2}}x + \rho).$$

Now

$$(21)\quad Y(p, \theta, t) = 1 + it\theta^{-\frac{1}{2}}V(p, \theta) - (t^2/2)\theta^{-1}M(p, \theta) + \gamma(t^3/3!)\theta^{-3/2}E[|p_{n+1} - p_n|^3 \mid p_n = p]$$

where

$$(22)\quad M(p, \theta) = E[(p_{n+1} - p_n)^2 \mid p_n = p] = \theta^2 N(p),$$

$$(23)\quad N(p) = (1 - p)^2\varphi(p) + \zeta^2 p^2(1 - \varphi(p)),$$

and $|\gamma| \leqq 1$. Expanding W and N around $p = \rho$ and noting that

$$E[|p_{n+1} - p_n|^3 \mid p_n = p] \leqq \max(1, \zeta^3)\theta^3$$

we obtain

$$(24)\quad Y(\theta^{\frac{1}{2}}x + \rho, \theta, t) = 1 + itx\theta W'(\rho) - (t^2/2)\theta N(\rho) + it\theta^{3/2}(x^2/2)W''(p^*) - (t^2/2)\theta^{3/2}xN'(p') + t^3O(\theta^{3/2})$$

where p^* and p' are points in the unit interval. By Lemma 1 $\int_{-\infty}^{\infty} x^2\, dG_\theta(x)$ and thus also $\int_{-\infty}^{\infty} |x|\, dG_\theta(x)$ are bounded functions of θ. Thus substituting (24) into (19), cancelling the first term on the right and the term on the left, and dividing what remains by $t\theta$ (assuming $t \neq 0$) we obtain

$$(25)\quad W'(\rho) \int_{-\infty}^{\infty} e^{ixt} ix\, dG_\theta(x) - (t/2)N(\rho) \int_{-\infty}^{\infty} e^{ixt}\, dG_\theta(x) = O(\theta^{\frac{1}{2}}).$$

As a consequence of Lemma 1 the family G_θ is completely compact, and $|e^{ixt}ix| = |x|$ is uniformly integrable with respect to G_θ. Suppose that $\theta_k \to 0$ and G_{θ_k} converges completely to a distribution function G as $k \to \infty$. Then

$$(26)\quad \int_{-\infty}^{\infty} |x|\, dG(x) < \infty,$$

and taking $\theta = \theta_k$ in (25) and letting $k \to \infty$ we obtain

$$(27)\quad W'(\rho) \int_{-\infty}^{\infty} e^{ixt} ix\, dG(x) = (t/2)N(\rho) \int_{-\infty}^{\infty} e^{ixt}\, dG(x)$$

for $t \neq 0$. Since, as a consequence of (26), both sides are continuous in t, (27) must hold for all real t. Equation (26) permits us to rewrite (27) in the form

$$(d/dt) \int_{-\infty}^{\infty} e^{ixt}\, dG(x) = -t\sigma^2 \int_{-\infty}^{\infty} e^{ixt}\, dG(x)$$

where $\sigma^2 = -N(\rho)/2W'(\rho) = N(\rho)/2\,|W'(\rho)|$. The only characteristic function satisfying this differential equation is

$$\int_{-\infty}^{\infty} e^{ixt}\, dG(x) = \exp\,[-(t^2/2)\sigma^2].$$

Thus $G(x) = \Phi(x/\sigma)$. From this it follows that $G_\theta(x) \to \Phi(x/\sigma)$ for all x as $\theta \to 0$, as was to be shown.

In the special case $\varphi(p) \equiv \rho$, $0 < \rho < 1$, and $\zeta = 1$ (i.e., $\theta_1 = \theta_2 = \theta$) the limiting distribution $F_{\theta,\theta,\varphi}$ of p_n as $n \to \infty$ is the same as the distribution of

$$(28)\quad S = \sum_{j=0}^{\infty} \epsilon_j \theta(1 - \theta)^j$$

where the ϵ_j are independent random variables with

$$(29)\quad P[\epsilon_j = 1] = \rho \quad \text{and} \quad P[\epsilon_j = 0] = 1 - \rho.$$

This was first observed by Kemeny and Snell [6]. Consequently, $F_{\theta,\theta,\varphi}(\theta^{\frac{1}{2}}x + \rho)$ is the distribution function of

$$(30)\quad (S - \rho)/\theta^{\frac{1}{2}} = \sum_{j=0}^{\infty} \tau_{\theta,j}$$

where $\tau_{\theta,j} = (\epsilon_j - \rho)\theta^{\frac{1}{2}}(1 - \theta)^j$. Noting that $\tau_{\theta,0}, \tau_{\theta,1}, \tau_{\theta,2}, \cdots$ are independent with

$$E[\tau_{\theta,j}] = 0,$$

$$E[\tau_{\theta,j}^2] = \rho(1 - \rho)\theta(1 - \theta)^{2j}, \qquad \text{and}$$

$$E[|\tau_{\theta,j}|^3] = \rho(1 - \rho)[(1 - \rho)^2 + \rho^2]\theta^{3/2}(1 - \theta)^{3j}$$

so that

$$\sum_{j=0}^{\infty} E[\tau_{\theta,j}^2] = \rho(1-\rho)/(2-\theta) \rightarrow \rho(1-\rho)/2 \qquad \text{as} \quad \theta \rightarrow 0$$

and

$$\sum_{j=0}^{\infty} E[|\tau_{\theta,j}|^3] = \rho(1-\rho)[(1-\rho)^2+\rho^2]\theta^{\frac{1}{2}}/[1+(1-\theta)+(1-\theta)^2] \rightarrow 0 \quad \text{as} \quad \theta \rightarrow 0$$

the possibility is strongly suggested that the result of Theorem 1 can be obtained in this case by a minor modification of a standard proof of the classical Liapounov theorem. This is indeed the case. We omit the details since the method developed in the proof of Theorem 1 can be applied to a much broader range of problems of the type considered in this paper.

3. Lemma 2 and Theorem 2. If $\theta_2 > 0$ is held fixed and θ_1 is permitted to approach 0, the distribution $dF_{\theta_1,\theta_2,\varphi}$ obviously concentrates at 0, and it is intuitively clear that a limiting distribution obtained by suitable normalization would be positively skewed. Thus we expect the limiting behavior of the distributions $dF_{\theta_1,\theta_2,\varphi}$ in this case to differ radically from that obtained in Section 2. The full extent of the discrepancy between the two cases is revealed by the following theorem.

THEOREM 2. *If* (1)–(4) *hold and* $\theta_2 > 0$, *then*

$$\lim_{\theta_1 \rightarrow 0} F_{\theta_1,\theta_2,\varphi}(\theta_1 x) = \mathop{*}_{n=0}^{\infty} G_{\varphi(0)}(x/(1-\theta_2)^n)$$

where $G_\omega(y)$ *is the geometric distribution with saltus* $(1-\omega)\omega^k$ *at* $y = k$, $k = 0, 1, 2, \cdots$ *and* $*$ *denotes convolution.*

The proof of this theorem parallels that of Theorem 1, but is quite different in many respects. First we require a lemma, analogous to Lemma 1, to the effect that the normalization of $F_{\theta_1,\theta_2,\varphi}$ indicated in the statement of Theorem 2 is correct. Note that neither the lemma nor the theorem of this section imposes a restriction on φ' comparable to (10).

LEMMA 2. *Under the hypotheses of Theorem* 2

$$\int_{-\infty}^{\infty} p^2\, dF_{\theta_1,\theta_2,\varphi}(p) = O(\theta_1^2).$$

PROOF. Taking $\rho = 0$ in (15) we obtain

$$E[p_{n+1}^2] = E[p_n^2] + 2E[p_n V^*(p_n, \theta_1)] + E[M^*(p_n, \theta_1)] \tag{31}$$

where

$$V^*(p_n, \theta_1) = E[p_{n+1} - p_n \mid p_n] = \theta_1(1-p_n)\varphi(p_n) - \theta_2 p_n(1-\varphi(p_n)) \tag{32}$$

and

$$M^*(p_n, \theta_1) = E[(p_{n+1}-p_n)^2 \mid p_n] = \theta_1^2(1-p_n)^2\varphi(p_n) + \theta_2^2 p_n^2(1-\varphi(p_n)). \tag{33}$$

Letting $n \to \infty$ in (31) yields

(34) $$0 = 2 \int_0^1 pV^*(p, \theta_1)\, dF^*_{\theta_1}(p) + \int_0^1 M^*(p, \theta_1)\, dF^*_{\theta_1}(p)$$

where $F^*_{\theta_1} = F_{\theta_1, \theta_2, \varphi}$ or

(35) $$\int_0^1 \theta_2 p^2(2 - \theta_2)(1 - \varphi(p))\, dF^*_{\theta_1}(p) = 2 \int_0^1 p\theta_1(1 - p)\varphi(p)\, dF^*_{\theta_1}(p) + \int_0^1 \theta_1^2(1 - p)^2\varphi(p)\, dF^*_{\theta_1}(p).$$

Thus

(36) $$\theta_2(1 - \epsilon_2) \int_0^1 p^2\, dF^*_{\theta_1}(p) \leqq 2\theta_1\epsilon_2\left(\int_0^1 p^2\, dF^*_{\theta_1}(p)\right)^{\frac{1}{2}} + \theta_1^2\epsilon_2$$

from which Lemma 2 follows easily.

Proof of Theorem 2. Paralleling the derivation of (19) we obtain

(37) $$\int_{-\infty}^{\infty} e^{ixt}\, dH_{\theta_1}(x) = \int_{-\infty}^{\infty} e^{ixt}Y^*(\theta_1 x, \theta_1)\, dH_{\theta_1}(x)$$

where $H_{\theta_1}(x) = F^*_{\theta_1}(\theta_1 x)$ and

(38) $$\begin{aligned} Y^*(p, \theta_1) &= E[\exp(i\theta_1^{-1}(p_{n+1} - p_n)t) \mid p_n = p] \\ &= \exp(it(1 - p))\varphi(p) + \exp(-it\theta_1^{-1}\theta_2 p)(1 - \varphi(p)) \end{aligned}$$

so that

(39) $$\begin{aligned} Y^*(\theta_1 x, \theta_1) &= e^{it(1-\theta_1 x)}\varphi(\theta_1 x) + e^{-it\theta_2 x}(1 - \varphi(\theta_1 x)) \\ &= e^{it}\varphi(0) + e^{-it\theta_2 x}(1 - \varphi(0)) + O(\theta_1 x). \end{aligned}$$

Substituting (39) into (37) and noting that $\int_{-\infty}^{\infty} |x|\, dH_{\theta_1}(x)$ is a bounded function of θ_1 as a consequence of Lemma 2, we find that

(40) $$\int_{-\infty}^{\infty} e^{ixt}\, dH_{\theta_1}(x) = e^{it}\varphi(0) \int_{-\infty}^{\infty} e^{ixt}\, dH_{\theta_1}(x) + (1 - \varphi(0)) \int_{-\infty}^{\infty} e^{ix(1-\theta_2)t}\, dH_{\theta_1}(x) + O(\theta_1).$$

It follows that H_{θ_1} converges as $\theta_1 \to 0$ to the distribution function H whose characteristic function φ_H satisfies the functional equation

(41) $$\varphi_H(t) = [1 - \varphi(0)]\varphi_H((1 - \theta_2)t)/[1 - \varphi(0)e^{it}],$$

i.e., $\varphi_H(t) = \prod_{n=0}^{\infty} [1 - \varphi(0)]/[1 - \varphi(0) \exp(i(1 - \theta_2)^n t)]$. Thus as $\theta_1 \to 0$

$$H_{\theta_1}(x) \to H(x) = \mathop{*}_{n=0}^{\infty} G_{\varphi(0)}(x/(1 - \theta_2)^n)$$

as claimed.

By applying Theorem 2 to the random walk $1 - p_1, 1 - p_2, 1 - p_3, \cdots$ we easily obtain

$$\lim_{\theta_2 \to 0} 1 - F_{\theta_1, \theta_2, \varphi}(1 - \theta_2 x) = \mathop{*}_{n=0}^{\infty} G_{1-\varphi(1)}(x/(1 - \theta_1)^n)$$

for $\theta_1 > 0$.

Since H is an infinite convolution of purely discrete distributions, it is either purely discrete or singular or absolutely continuous (Jessen and Wintner [4]), and the first possibility is precluded in this case by a theorem of Lévy [7].

As in the case of Theorem 1, a special method of proof is available for Theorem 2 when $\varphi(p) \equiv \rho$ where $0 < \rho < 1$. As this alternative derivation sheds light on the origin of the geometric distributions from which H is built up, we sketch it below. Let ϵ_j be a sequence of independent random variables satisfying (29) and denote θ_2 by θ_0 for the time being. It is easily shown that the random variable

$$T = \theta_1 \sum_{j=0}^{\infty} \epsilon_j \prod_{k=0}^{j-1} (1 - \theta_{\epsilon_k})$$

has the distribution function $F_{\theta_1,\theta_0,\varphi}$ so that $T^* = T/\theta_1$ has the distribution function H_{θ_1}. But

$$\lim_{\theta_1 \to 0} T^* = \sum_{j=0}^{\infty} \epsilon_j (1 - \theta_0)^{\sum_{l=0}^{j-1}(1-\epsilon_l)}$$

with probability 1

$$= \sum_{n=0}^{\infty} (1 - \theta_0)^n X_n \tag{42}$$

where

X_0 = number of 1's before the first 0 of $\{\epsilon_j\}$, and

X_n = number if 1's after the nth and before the $(n + 1)$st 0 of $\{\epsilon_j\}$, $n \geqq 1$.

The X_k's are independently and identically distributed with the geometric distribution G_ρ. It follows that H_{θ_1} converges to the distribution function H of the random variable (42) as $\theta_1 \to 0$.

4. Theorems 3 and 4. Let $f(p, \theta)$ be a function of two variables defined on a rectangle $R = [0, 1] \times [0, \delta]$, $\delta > 0$, such that

$$f \in C^2(R), \tag{43}$$

$$0 \leqq f(p, \theta) \leqq 1 \quad \text{throughout} \quad R, \tag{44}$$

$$f(p, \theta) > p \quad \text{for} \quad 0 \leqq p < 1, \quad 0 < \theta \leqq \delta, \quad \text{and} \tag{45}$$

$$f(1, \theta) = 1 \quad \text{for} \quad 0 \leqq \theta \leqq 1. \tag{46}$$

Returning to the schematic description of a learning experiment given in the introduction, suppose that if E_1 occurs on trial n, p_n increases to $p_{n+1} = f(p_n, \theta_1)$, while if E_2 occurs on trial n, $1 - p_n$ increases to $1 - p_{n+1} = f(1 - p_n, \theta_2)$ so that p_n decreases to $p_{n+1} = 1 - f(1 - p_n, \theta_2)$. In this way the function f determines a two-parameter family of learning models for the experiment. We will subject f to further axioms sufficient to obtain the asymptotic behavior of the preceding two sections. To this end we require, informally speaking, that the second variable of f be a "learning rate" parameter as is θ in the linear models for which $f(p, \theta) = p + \theta(1 - p)$. Thus the increment in p following an occurrence of E_1 should be an increasing function of θ_1 with $\theta_1 = 0$ corresponding to no increment. More precisely, we require that

(47) $$(\partial/\partial\theta)f(p, 0) > 0, \qquad 0 \leqq p < 1,$$

and

(48) $$f(p, 0) = p, \qquad 0 \leqq p \leqq 1.$$

It is also natural to stipulate that f be strictly increasing in p,

(49) $$(\partial/\partial p)f(p, \theta) > 0 \quad \text{throughout} \quad R.$$

Finally we require that

(50) $$(\partial/\partial p)f(p, \theta) < 1, \qquad 0 < \theta \leqq \delta, \quad 0 \leqq p \leqq 1.$$

This implies that $f(\cdot, \theta)$ for $\theta > 0$ is *strictly distance diminishing* in the sense that, for some $\gamma_\theta < 1$ and all $0 \leqq p \leqq 1$, $|f(p, \theta) - f(p', \theta)| \leqq \gamma_\theta |p - p'|$. This property, in conjunction with (43), (3), and (4) insures that the stochastic process $\{p_n\}$ associated with f, $\theta_1 > 0$, $\theta_2 > 0$, and φ has a limiting distribution $F_{\theta_1,\theta_2,\varphi,f}$ as $n \to \infty$. The argument given by Karlin ([5], Section 6) for linear models requires very little modification.

We can now assert the following:

THEOREM 3. *If f satisfies* (43)–(50), *and φ satisfies* (3) *and* (4), *then for* $\theta_2 > 0$

$$\lim_{\theta_1 \to 0} F_{\theta_1,\theta_2,\varphi,f}(\theta_1 x) = \mathop{*}_{n=0}^{\infty} G_{\varphi(0)}(x/ab^n)$$

where $a = (\partial/\partial\theta)f(0, 0) > 0$ *and* $0 < b = (\partial/\partial p)f(1, \theta_2) < 1$.

PROOF. The proofs in Sections 2 and 3 carry over almost immediately to the theorems of this section so we need not present many details here. Writing $u(p, \theta) = f(p, \theta) - p$, we obtain the expressions

(51) $$f(p, \theta) = p + u(p, \theta)$$
$$1 - f(1 - p, \theta) = p - u(1 - p, \theta)$$

which imitate the expressions $f(p, \theta) = p + \theta(1 - p)$ and $1 - f(1 - p, \theta) = p - \theta p$ for the linear model and can be put to the same use in much mathematical work. Equations (25) and (26), for instance, go over into

(52) $$V^*(p_n, \theta_1) = u(p_n, \theta_1)\varphi(p_n) - u(1 - p_n, \theta_2)(1 - \varphi(p_n))$$

and

(53) $$M^*(p_n, \theta_1) = u^2(p_n, \theta_1)\varphi(p_n) + u^2(1 - p_n, \theta_2)(1 - \varphi(p_n)).$$

Using the Taylor expansions

$$u(1 - p, \theta_2) = -p(\partial/\partial p)u(p^*, \theta_2), \qquad 1 - p \leqq p^* \leqq 1$$

and

$$u(p, \theta_1) = \theta_1(\partial/\partial\theta)u(p, \theta^*), \qquad 0 < \theta^* < \theta_1$$

in conjunction with (34) and (50) (from which it follows that

$$(\partial/\partial p)u(p^*, \theta) < 0)$$

and defining $F_{\theta_1}^*$ in an obvious way, we obtain

$$\begin{aligned} \int_0^1 p^2 \,|(\partial/\partial p)u(p^*, \theta_2)|(2 - |(\partial/\partial p)u(p^*, \theta_2)|)(1 - \varphi(p))\, dF_{\theta_1}^*(p) \\ = 2\theta_1 \int_0^1 p(\partial/\partial\theta)u(p, \theta^*)\varphi(p)\, dF_{\theta_1}^*(p) \\ + \theta_1^2 \int_0^1 ((\partial/\partial\theta)u(p, \theta^*))^2\varphi(p)\, dF_{\theta_1}^*(p) \end{aligned} \tag{54}$$

in place of (35). From (43), (49), and (50) we obtain

$$\min_{0 \leqq p \leqq 1} |(\partial/\partial p)u(p, \theta_2)| > 0 \quad \text{and} \quad \max_{0 \leqq p \leqq 1} |(\partial/\partial p)u(p, \theta_2)| < 1$$

which, in combination with (54), yields an analogue of (36). Thus the conclusion of Lemma 2 holds with $F_{\theta_1, \theta_2, \varphi}$ replaced by $F_{\theta_1, \theta_2, \varphi, f}$. This permits the rest of the proof of Theorem 2 to be carried over directly even though we now have

$$\begin{aligned} Y^*(\theta_1 x, \theta_1) &= \exp[it\theta_1^{-1}u(\theta_1 x, \theta_1)]\varphi(x\theta_1) \\ &\quad + \exp[-it\theta_1^{-1}u(1 - \theta_1 x, \theta_2)](1 - \varphi(x\theta_1)) \\ &= \exp[it(\partial/\partial\theta)u(0, 0)]\varphi(0) + \exp[itx(\partial/\partial p)u(1, \theta_2)](1 - \varphi(0)) \\ &\quad + O(\theta_1) + O(\theta_1 x) + O(\theta_1 x^2) \end{aligned}$$

instead of (39).

The comparable result for $\theta_2 \to 0$ while $\theta_1 > 0$ is fixed is

$$\lim_{\theta_2 \to 0} 1 - F_{\theta_1, \theta_2, \varphi, f}(1 - \theta_2 x) = \mathop{*}_{n=0}^{\infty} G_{1-\varphi(1)}(x/a\tilde{b})$$

where $\tilde{b} = (\partial/\partial p)f(1, \theta_1)$.

I have found it necessary to make further assumptions in order to prove the analogue of Theorem 1 within the framework of this section. First, my analysis requires that f be slightly smoother than was required previously. Specifically it is assumed that

$$f \in C^3(R). \tag{55}$$

Note that as a consequence of (48) the distance diminishing property (50) is lost in the limit as $\theta \to 0$, i.e., $(\partial/\partial p)f(p, 0) = 1$ for all $0 \leqq p \leqq 1$. Our second new assumption is that this loss does not occur too quickly, that is,

$$\lim_{\theta \to 0} [(\partial/\partial p)f(p, \theta) - 1]/\theta = (\partial^2 f/\partial\theta\partial p)(p, 0) < 0, \quad 0 \leqq p \leqq 1. \tag{56}$$

For the linear models for instance $\partial^2 f/\partial\theta\partial p \equiv -1$.

We can now state and prove the following generalization of Theorem 1.

THEOREM 4. *Suppose that f satisfies* (44)–(50), (55), *and* (56), *while φ satisfies* (3), (4), *and*

$$\varphi'(p) < \frac{\varphi(p)\left|\frac{\partial^2}{\partial\theta\partial p} f(p, 0)\right| + (1 - \varphi(p))\left|\frac{\partial^2}{\partial\theta\partial p} f(1 - p, 0)\right|\zeta}{\frac{\partial}{\partial\theta} f(p, 0) + \frac{\partial}{\partial\theta} f(1 - p, \theta)\zeta} \tag{57}$$

for all $0 \leqq p \leqq 1$ *where* $\zeta > 0$. *Then the equation*

$$(\partial/\partial\theta)E[p_{n+1} - p_n \mid p_n]\,|_{\theta=0} = 0 \tag{58}$$

has a unique root $p_n = \rho = \rho_{\zeta,\varphi,f}$ *in* $(0, 1)$ *and*

$$F_{\theta,\zeta\theta,\varphi,f}(\theta^{\frac{1}{2}}x + \rho) \rightarrow \Phi(x/\sigma)$$

where

$$\sigma^2 = (\partial^2/\partial\theta^2)M(\rho, 0)/4|(\partial^2/\partial p\partial\theta)V(\rho, 0)|.$$

$M(p, \theta)$ *is given by* (61), *and* $V(p, \theta)$ *by* (59) *below.*

PROOF. Defining $V(p, \theta)$ as in (12), we have

$$V(p, \theta) = \varphi(p)u(p, \theta) - (1 - \varphi(p))u(1 - p, \theta\zeta). \tag{59}$$

In view of (57) the argument on W at the beginning of the proof of Lemma 1 can be applied directly to $(\partial/\partial\theta)V(p, 0)$ to yield the existence and uniqueness of the root ρ of (58).

Much as in the proof of Lemma 1 we obtain (16) where F_θ now is $F_{\theta,\zeta\theta,\varphi,f}$. Writing

$$V(p, \theta) = \theta(p - \rho)(\partial^2V/\partial p\partial\theta)(p^*, 0) + O(\theta^2)$$

where p^* is between p and ρ and O is uniform in p we obtain

$$-\int_{-\infty}^{\infty} (p - \rho)^2(\partial^2V/\partial p\partial\theta)(p^*, 0)\, dF_\theta(p) = O(\theta).$$

But $\sup_{0 \leqq p \leqq 1} (\partial^2V/\partial\theta\partial p)(p, 0) < 0$ as a consequence of (3), (55), and (57), so this gives

$$\int_{-\infty}^{\infty} (p - \rho)^2\, dF_\theta(p) = O(\theta), \tag{60}$$

the conclusion of Lemma 1.

Defining $M(p, \theta)$ by the first equality in (22) , i.e.,

$$M(p, \theta) = u^2(p, \theta)\varphi(p) + u^2(1 - p, \zeta\theta)(1 - \varphi(p)), \tag{61}$$

$Y(p, \theta, t)$ by (18), and G_θ by (20), we have (19) and (21) just as in the proof of Theorem 1. Again we have

$$E[|p_{n+1} - p_n|^3 \mid p_n = p] = O(\theta^3)$$

uniformly in p. Substituting this and the expansions

$$V(p, \theta) = \theta(p - \rho)(\partial^2/\partial p\partial\theta)V(\rho, 0) + \theta^2O([(p - \rho)\theta^{-\frac{1}{2}}]^2) + O(\theta^2)$$

and

$$M(p, \theta) = \theta^2(\partial^2/\partial\theta^2)M(\rho, 0)/2 + \theta^{5/2}O(|(p - \rho)\theta^{-\frac{1}{2}}|) + O(\theta^3)$$

($O(\theta^2)$ and $O(\theta^3)$ are uniform in p) into (19) and using (60) we obtain after some manipulation

$$(\partial^2/\partial p\partial\theta)V(\rho, 0)\int_{-\infty}^{\infty} e^{itx}ix\, dG_\theta(x) - (t/4)(\partial^2/\partial\theta^2)M(\rho, 0)\int_{-\infty}^{\infty} e^{itx}\, dG_\theta(x) = O(\theta^{\frac{1}{2}}) \tag{62}$$

for $t \neq 0$. From this point the argument proceeds just like that following (25) in the proof of Theorem 1. The quantity $(\partial^2/\partial\theta^2)M(\rho, 0)$ is positive as a consequence of (47).

5. Theorem 5. Consider a one-parameter family of finite Markov chains $\{\chi^{(N)}\}_{N=1}^{\infty}$ where $\chi^{(N)} = X_1^{(N)}, X_2^{(N)}, X_3^{(N)}, \cdots$ has state space $\{0, 1, \cdots, N\}$ and transition probabilities

$$(63) \qquad P[X_{n+1}^{(N)} = X_n^{(N)} + 1 \mid X_n^{(N)}] = \varphi(X_n^{(N)}/N),$$

$$(64) \qquad P[X_{n+1}^{(N)} = X_n^{(N)} - 1 \mid X_n^{(N)}] = \psi(X_n^{(N)}/N),$$

$$(65) \qquad P[X_{n+1}^{(N)} = X_n^{(N)} \mid X_n^{(N)}] = 1 - \varphi(X_n^{(N)}/N) - \psi(X_n^{(N)}/N)$$

which depend on the relative position of the present state within the state space. Putting $p_n = p_n^{(N)} = X_n^{(N)}/N$, these equations yield the transition probabilities (7), (8), and (9) for a corresponding chain $\mathcal{P}^{(N)} = p_1^{(N)}, p_2^{(N)}, p_3^{(N)}, \cdots$ with state space $\{0, 1/N, \cdots, (N-1)/N, 1\}$ where now φ and ψ, instead of having the specific form which arises in the N-element pattern model, are subject only to the following restrictions:

$$(66) \qquad 1 \geqq \varphi(p) > 0 \qquad \text{for all } 0 \leqq p < 1, \varphi(1) = 0,$$

$$(67) \qquad 1 \geqq \psi(p) > 0 \qquad \text{for all } 0 < p \leqq 1, \psi(0) = 0,$$

$$(68) \qquad 1 \geqq \varphi(p) + \psi(p) \qquad \text{for all } 0 \leqq p \leqq 1, 1 > \varphi(p_0) + \psi(p_0) \quad \text{for some } 0 \leqq p_0 \leqq 1,$$

$$(69) \qquad \varphi, \psi \in C^2([0, 1]), \qquad \text{and}$$

$$(70) \qquad \varphi'(p) < \psi'(p) \qquad \text{for all } 0 \leqq p \leqq 1.$$

Equations (66) and (67) insure that $\mathcal{P}^{(N)}$ is irreducible. By (69) and the second condition in (68) there is a subinterval I of $[0, 1]$ of positive length ϵ such that $1 > \varphi(p) + \psi(p)$ for $p \in I$. For $N > 1/\epsilon$ the chain $\mathcal{P}^{(N)}$ has a state which belongs to I, hence $\mathcal{P}^{(N)}$ is aperiodic. Thus

$$(71) \qquad \lim_{n\to\infty} P[p_n^{(N)} \leqq x] = F_N(x)$$

exists for all x and is independent of the distribution of $p_1^{(N)}$. The condition (70) (which is satisfied, for instance, in the interesting case $\varphi'(p) < 0$, $\psi'(p) > 0$, $0 \leqq p \leqq 1$) together with (66) and (67) implies that the equation

$$(72) \qquad \varphi(\rho) = \psi(\rho)$$

has a unique root ρ in $(0, 1)$.

We can now state the following theorem.

THEOREM 5. *If Conditions* (66)–(70) *hold, then*

$$\lim_{N\to\infty} F_N(N^{-\frac{1}{2}}y + \rho) = \Phi(y/\sigma)$$

where $\sigma^2 = (\varphi(\rho) + \psi(\rho))/2(\psi'(\rho) - \varphi'(\rho))$. *Equivalently,*

$$\lim_{N\to\infty} \lim_{n\to\infty} P[(X_n^{(N)} - N\rho)/N^{\frac{1}{2}}\sigma \leqq y] = \Phi(y).$$

The proof is quite similar to that of Theorem 1, though somewhat simpler, so we omit the details.

REFERENCES

[1] Bush, R. R. and Mosteller, F. (1955). *Stochastic Models for Learning.* Wiley, New York.

[2] Estes, W. K. (1959). Component and pattern models with Markovian interpretations. In R. R. Bush and W. K. Estes (Eds.), *Studies in Mathematical Learning Theory.* 9–52. Stanford Univ. Press.

[3] Estes, W. K. and Suppes, P. (1959). Foundations of linear models. In R. R. Bush and W. K. Estes (Eds.), *Studies in Mathematical Learning Theory.* 137–179. Stanford Univ. Press.

[4] Jessen, B. and Wintner, A. (1935). Distribution functions and the Riemann zeta function. *Trans. Amer. Math. Soc.* **38** 48–88.

[5] Karlin, S. (1953). Some random walks arising in learning models I. *Pacific J. Math.* **3** 725–756.

[6] Kemeny, J. G. and Snell, J. L. (1957). Markov processes in learning theory. *Psychometrika* **22** 221–230.

[7] Lévy, P. (1931) Sur les séries dont les termes sont des variables éventuelles indépendantes. *Studia Math.* **3** 119–155.

[8] McGregor, J. R. and Hui, Y. Y. (1962). Limiting distributions associated with certain stochastic learning models. *Ann. Math. Statist.* **33** 1281–1285.

[9] McGregor, J. R. and Zidek, J. V. (1965). Limiting distributions of response probabilities. *Ann. Math. Statist.* **36** 706–707.