SEQUENTIAL COUNTERBALANCING IN LATIN SQUARES

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A $k \times k$ Latin square is an arrangement of k types into a k-order matrix, such that each type occurs once in each row and in each column. For a general discussion of Latin squares and orthogonal Latin squares see Mann [3].

In experiments involving k successive treatments on nk subjects it is often desirable to control progressive effects by a Latin square design. Here the Latin square represents not a fractional selection of k^2 treatment combinations from a universe of k^3, but a selection of k sequences of treatments from a universe of $k!$ permutations. Since residual effects from prior treatments often affect responses, it is desirable that such squares be sequentially counterbalanced for immediate residual effects. By this is meant that within the rows of the square every treatment immediately precedes every other treatment an equal number of times.

It is well known that a Latin square of order $k = p' - 1$ whose elements $L_{i,j}$ are of the form $(i \cdot j)(\text{mod } p')$ will be sequentially counterbalanced, where p' is a prime and i and j, the row and column indices, range from 1 to k. A proof is offered by Alimena [1], who seems unaware that his construction is a permutation by columns of a modular multiplication table, having identical sequential properties. A more general construction is offered by Bradley [2], which satisfies Theorem 1 below for any $k \equiv 0 \pmod 2$.

For L any Latin square, let $L_{i,j}$ be the cell occurring in row i, column j, where i and j range from 0 to $k - 1$.

DEFINITION. A Latin square is called cyclic if $L_{ij} = m$ implies $L_{i,j+q} = m + q$ for all i, j, q, where all values are reduced modulo k, as throughout this paper.

THEOREM 1. *A cyclic Latin square is sequentially counterbalanced if and only if in any row i the set of all values of $d(j)$ is a permutation of the first $k - 1$ natural numbers, for $d(j) = (s - r)$ where $L_{ij} = r$, $L_{i,j+1} = s$.*

PROOF OF NECESSITY. From the definition of a cyclic Latin square it follows that $d(j)$ is independent of i. Suppose L were a sequentially counterbalanced k-order cyclic Latin square such that $d(j) = d(j')$ for some $j \neq j'$. Then $L_{ij} = m$ is followed by $L_{i,j+1} = m + d(j)$. But in some row $i' \neq i$, $L_{i',j'} = m$ is followed by $L_{i',j'+1} = m + d(j') = m + d(j)$. Therefore in L, the successive types $(m; m + d(j))$ occur twice. But this is impossible, since there are $k(k - 1)$ ordered pairs of k different types, and $k(k - 1)$ different pairs of consecutive cells in L.

PROOF OF SUFFICIENCY. By the definition of a Latin square, m occurs in every one of the first $(k - 1)$ columns. Since $d(j) \neq d(j')$ for $j \neq j'$, it follows that m

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is followed by every one of the types $m + 1, m + 2, \cdots, m + k - 1$. This completes the proof of Theorem 1.

COROLLARY. *No sequentially counterbalanced cyclic Latin square exists of an order* $k \equiv 1 \pmod 2$.

PROOF. Let $k \equiv 1 \pmod 2 = 2c + 1$. If cyclic Latin square L of order k satisfied Theorem 1, then

$$\sum_{j=0}^{k-2} d(j) = \sum_{n=1}^{k-1} n = kc \equiv 0 \pmod k.$$

But if $L_{i,0} = m$, we have $L_{i,k-1} = m + kc = m$, which contradicts the definition of a Latin square.

DEFINITION. Two k-order Latin squares within whose rows all $k(k - 1)$ ordered pairs of k different types occur exactly twice are called complementary.

THEOREM 2. *For any* $k = 2c + 1$, *the k-order Latin squares* L' *and* L'', *where* $L'_{ij} = (-1)^j[(j + 1)/2] + i$, $L''_{ij} = (-1)^j[(k - j)/2] + i$, *are orthogonal and complementary. Here* $[x]$ *denotes the integral part of* x, *where* x *is not reduced modulo* k.

PROOF. L' and L'' are cyclic Latin squares by definition. Furthermore, $d(j)$ in L' assumes the values $2, 4, \cdots, k - 1$ twice. Hence every pair of types $(L'_{ij}; L'_{i,j+1})$ is of the form $(m; m + 2y)$ for some natural number y. It follows that kc different ordered pairs occur twice in L'. Similarly, L'' contains kc pairs of the form $(m; m + 2y + 1)$, since $d(j)$ in L'' assumes c different odd values twice in any row. Since $(m; m + 2y) \neq (m; m + 2y + 1)$, it follows that $kc + kc = k(k - 1)$ different pairs occur twice in L' and L'', so L' and L'' are complementary.

To demonstrate orthogonality, set

$$b(j) = j + c + 1 \qquad \text{if } j \text{ is even,}$$
$$= c - j \qquad \text{if } j \text{ is odd.}$$

Then $L'_{ij} - L''_{ij} \equiv b(j) \pmod k$. Now if $b(2t) \equiv b(2t' + 1) \pmod k$, then $2t + c + 1 \equiv c - 2t' - 1 \pmod k$, and $2t + 2t' + 2 \equiv 0 \pmod k$. Hence since $2t + 2t' + 2 < 2k$, we must have $2t + 2t' + 2 = k$, which is impossible. This completes the proof of Theorem 2.

An example of squares constructed by the rules in Theorem 2 follows, for $k = 7$:

$$\begin{matrix} 0&6&1&5&2&4&3\\ 1&0&2&6&3&5&4\\ 2&1&3&0&4&6&5\\ 3&2&4&1&5&0&6\\ 4&3&5&2&6&1&0\\ 5&4&6&3&0&2&1\\ 6&5&0&4&1&3&2 \end{matrix} = L'; \qquad \begin{matrix} 3&4&2&5&1&6&0\\ 4&5&3&6&2&0&1\\ 5&6&4&0&3&1&2\\ 6&0&5&1&4&2&3\\ 0&1&6&2&5&3&4\\ 1&2&0&3&6&4&5\\ 2&3&1&4&0&5&6 \end{matrix} = L''.$$

The extension of this construction to within-column counterbalancing is accomplished by permuting the last $(k - 1)$ rows so that the first column is the transpose of the first row for both squares. The construction will also give pairs of Greco-Latin squares counterbalanced for sequential effects within (but not between) alphabets.

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