AN ASYMPTOTICALLY DISTRIBUTION-FREE MULTIPLE COMPARISON PROCEDURE-TREATMENTS VS. CONTROL[1]

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1. Introduction and summary. Let X_{i0} and X_{ij} $(i = 1, \cdots, n; j = 1, \cdots, k)$ be the independent measurements on the control and jth treatment in the ith block, with $P(X_{ij} \leqq x) = F_j(x - b_i)$ Here b_i is the block i nuisance parameter and the F_j ; $j = 0, \cdots, k$, are assumed continuous. Nemenyi [5] suggests treatment-control comparisons based on the statistic $T = \max_j T_{0j}$ where T_{0j} is defined by (2.1). It is shown here that, under the null hypothesis

$$H_0 : F_j = F \text{ (unknown)}, \qquad j = 0, \cdots, k, \tag{1.1}$$

T is neither distribution-free for finite n, nor asymptotically distribution-free. We also modify Nemenyi's procedure so that it is asymptotically distribution-free.

2. Asymptotic distribution theory. Let $Y_{0j}^{(i)} = |X_{i0} - X_{ij}|$ and $R_{0j}^{(i)} =$ rank of $Y_{0j}^{(i)}$ in the ranking from least to greatest of $Y_{0j}^{(i)}$; $i = 1, \cdots, n$. Furthermore, let

$$T_{0j} = \textstyle\sum_{i=1}^{n} R_{0j}^{(i)} \xi_{0j}^{(i)}, \tag{2.1}$$

where $\xi_{0j}^{(i)} = 1$ if $X_{i0} < X_{ij}$ and 0 otherwise.

Lemma 1. (Tukey [7]). *If* $S_1 = \sum_{i=1}^{n}$ *(rank* $|X_i|)\psi_i$ *where* $\psi_i = 1$ *if* $X_i < 0$ *and* 0 *otherwise, and* $S_2 = \sum_{i \leqq j}^{n} \psi_{ij}$ *where* $\psi_{ij} = 1$ *if* $X_i + X_j < 0$ *and* 0 *otherwise, then* $S_1 = S_2$.

Lemma 2. *Assume* $0 < \int F_0 \, dF_j < 1$ *for* $j = 1, \cdots, k$, *let* $H_j = F_j {}^* F_j$ *and* $U_j = T_{0j}/\binom{n}{2}$. *Then the random variables* $n^{\frac{1}{2}}(U_j - \int H_0 \, dH_j)$ *have an asymptotic k-variate normal distribution.*

Proof. From Lemma 1 we rewrite T_{0u} as

$$T_{0u} = \textstyle\sum_{i<j}^{n} \psi_{0u}^{(i,j)} + \sum_{i=1}^{n} \xi_{0u}^{(i)}, \tag{2.2}$$

where $\psi_{0u}^{(i,j)} = 1$ if $X_{i0} - X_{iu} + X_{j0} - X_{ju} < 0$ and 0 otherwise. The joint asymptotic normality of the random variables $n^{\frac{1}{2}}[(\sum_{i<j} \psi_{0u}^{(i,j)}/\binom{n}{2})) - \int H_0 \, dH_u]$ is a consequence of Hoeffding's [3] U-statistic theorem. The result follows by noting that $p - \lim n^{\frac{1}{2}}(\sum_{i=1}^{n} \xi_{0u}^{(i)})/\binom{n}{2}) = 0$.

Theorem. *T is not distribution-free as the correlation coefficient* $\rho_0{}^n(F)$, *be-*

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tween T_{0u} *and* T_{0v}, $0 \neq u \neq v$, *is*

$$\rho_0^n(F) = [(n+1)(2n+1)]^{-1} [(24\lambda(F) - 6)n^2 + (48\mu(F) - 72\lambda(F) + 7)n + (48\lambda(F) - 48\mu(F) + 1)] \tag{2.3}$$

where

$$\mu(F) = P(X_1 < X_2 ; X_1 < X_5 + X_6 - X_7) \tag{2.4}$$

and

$$\lambda(F) = P(X_1 < X_2 + X_3 - X_4 ; X_1 < X_5 + X_6 - X_7) \tag{2.5}$$

when $X_1, X_2, \cdots, X_7$ *are* iid *according to* F.

PROOF. From (2.2) we have,

$$E_0(T_{0u}T_{0v}) = E_0\left(\sum_{i<j}^n \psi_{0u}^{(i,j)}\right)\left(\sum_{i<j}^n \psi_{0v}^{(i,j)}\right) + 2E_0\left(\sum_{i<j}^n \psi_{0u}^{(i,j)}\right)\left(\sum_{i=1}^n \xi_{0v}^{(i)}\right) + E_0\left(\sum_{i=1}^n \xi_{0v}^{(i)}\right)\left(\sum_{i=1}^n \xi_{0u}^{(i)}\right) = E_0(A_1) + 2E_0(A_2) + E_0(A_3), \tag{2.6}$$

where A_1, A_2, A_3 are defined by (2.6). Now,

$$E_0(A_1) = n(n-1)/6 + n(n-1)(n-2)\lambda(F) + n(n-1)(n-2)(n-3)/16, \tag{2.7}$$

$$E_0(A_2) = n(n-1)\mu(F) + n(n-1)(n-2)/8, \tag{2.8}$$

$$E_0(A_3) = n/3 + n(n-1)/4. \tag{2.9}$$

Combining (2.7), (2.8), (2.9), we obtain $E_0(T_{0u}T_{0v})$ and (2.3) follows.

We thus see that, under H_0, T has the same asymptotic distribution as that of the maximum of k equi-correlated normal random variables, each having the same mean and variance. (The well-known expressions $E_0(T_{0j}) = n(n+1)/4$, $\sigma_0^2(T_{0j}) = n(n+1)(2n+1)/24$, are easily obtained from (2.1).) However T is not distribution-free since the null correlation coefficient $\rho_0^n(F)$, between T_{0u} and T_{0v}, $0 \neq u \neq v$, depends on F (except for $n = 1$).

Corollary. $0 < \rho^*(F) \leqq \frac{1}{2}$, *where* $\rho^*(F) = \lim_n \rho_0^n(F)$.

PROOF. From (2.3) we immediately see that

$$\rho^*(F) = 12\lambda(F) - 3. \tag{2.10}$$

In [4], Lehmann proves the inequality, $6/24 < \lambda(F) \leqq 7/24$, and the bounds on $\rho^*(F)$ follow.

From (2.10) we see that T is not asymptotically distribution-free. Lehmann gives the three values of $\lambda(F)$:

F	Normal	Rectangular	Cauchy
$\lambda(F)$	.2902	.2909	.2879

For F exponential, $\lambda(F) = .2894$, and some values of $\mu(F)$ are:

F	Normal	Rectangular	Exponential
$\mu(F)$	.3075	.3083	.3056

In Table 2.1 we give some values of $\rho_0^n(F)$ for F normal, rectangular, and exponential.

TABLE 2.1

	n														
	1	2	3	4	5	6	7	8	9	10	15	20	25	50	∞
Norm.	.333	.384	.409	.424	.434	.441	.446	.450	.453	.456	.464	.469	.471	.477	.483
Rect.	.333	.387	.413	.429	.439	.446	.452	.456	.460	.462	.471	.476	.479	.484	.490
Exp.	.333	.378	.401	.415	.424	.431	.436	.440	.443	.446	.454	.458	.461	.467	.472

3. An asymptotically distribution-free procedure. For the problem of selecting (without regard to order) which treatments are better than the control, Nemenyi's procedure is to select treatment j ($j = 1, \cdots, k$) if

$$T_{0j} \geqq n(n+1)/4 + d_{(\alpha,k)}^{(n)}(n(n+1)(2n+1)/24)^{\frac{1}{2}}, \tag{3.1}$$

where $d_{(\alpha,k)}^{(n)}$ is chosen so that

$$P_0[T < n(n+1)/4 + d_{(\alpha,k)}^{(n)})(n(n+1)(2n+1)/24)^{\frac{1}{2}}] \simeq 1 - \alpha. \tag{3.2}$$

Based on Monte Carlo evidence that the null correlation between T_{0u} and T_{0v} was close to $\frac{1}{2}$, and the assumption that T was distribution-free, Nemenyi proposed obtaining $d_{(\alpha,k)}^{(n)}$ from Dunnett's [1] multivariate t-tables ($n = \infty$). In view of (2.3), we propose obtaining $d_{(\alpha,k)}^{(n)}$ from Gupta's [2] tables of the equicorrelated multivariate normal distribution. The tables should be entered at $\rho = \hat{\rho}$ where $\hat{\rho}$ is a consistent estimate of $\rho^*(F)$. This procedure is asymptotically distribution-free. In [4], Lehmann proposed the unbiased and consistent estimate $\hat{\lambda}(F)$ of $\lambda(F)$, where $\hat{\lambda}$ (F) is the proportion of sextuples $(\alpha, \beta, \gamma; u, v, w)$ satisfying the simultaneous inequalities $(X_{\alpha u} < X_{\beta u} + X_{\alpha v} - X_{\beta v}\ ;\ X_{\alpha u} < X_{\gamma u} + X_{\alpha w} - X_{\gamma w})$. In the same manner, unbiased estimates of $\mu(F)$ could be used to provide unbiased estimates of $\rho_0^n(F)$. However, since the approach to $\rho^*(F)$ is rapid, the consistent estimate $\hat{\rho} = -3 + 12\hat{\lambda}(F)$ will suffice. Also, as Lehmann mentions, $\hat{\lambda}(F)$ is computationally tedious and in practice an estimate based on a small subset of the sextuples should be used.

4. Comments. The values given for $\rho^*(F)$ were all close to the upper bound and it seems that there should exist a better lower bound for $\lambda(F)$. For example, if (as one might intuitively feel) $\rho_0^n(F)$ is an increasing function of n for each F, then we could conclude that $\rho^*(F) > \frac{1}{3}$ and $\lambda(F) > 5/18$. At any rate, as Nemenyi and Steel [6] have pointed out, the dependence of the distribution of the maximum of several equi-correlated unit normal random variables on the common correlation is slight. (The reader can verify this by browsing through Gupta's tables.) In view of this fact, Nemenyi's procedure is "nearly" distribution-free and will be useful for comparisons of this nature.

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