A NOTE ON RECURRENCE RELATIONS BETWEEN EXPECTED VALUES OF FUNCTIONS OF ORDER STATISTICS

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1. Summary. In this note, some recurrence relations are derived between the expected values of functions of order statistics from any arbitrary distribution with continuous cumulative distribution function (cdf). These recurrence relations are closely related to some results obtained by Srikantan [2].

2. Some recurrence relations. Suppose X has an arbitrary distribution with a continuous cdf $F(x)$ and $h(\cdot)$ is a specified function such that $Eh(X)$ exists. Let $X_{k,n}$ denote the kth $(1 \leqq k \leqq n)$ order statistic in a random sample of size n from the distribution with cdf $F(x)$. Then we have the following theorem:

THEOREM. *For* $1 \leqq k \leqq m \leqq n$,

$$E\{h(X_{k,m})\} = \binom{m}{k} \sum_{s=0}^{i} (-1)^s (k/(k-i))\left[\binom{i}{s} \Big/ \binom{m-i+s}{k-i}\right] E\{h(X_{k-i,m-i+s})\}, \quad 0 \leqq i \leqq k-1, \tag{2.1}$$

and also

$$E\{h(X_{k,m})\} = \binom{m}{k} \sum_{s=0}^{j} (-1)^s (k/(k+s))\left[\binom{j}{s} \Big/ \binom{m-j+s}{k+s}\right] E\{h(X_{k+s,m-j+s})\}, \quad 0 \leqq j \leqq m-k. \tag{2.2}$$

PROOF. Let $G(u) = 1 - F(u)$ for all u and let $\int$ denote the integral over the entire real line. Then for $1 \leqq k \leqq m \leqq n$ and $0 \leqq i \leqq k-1$,

$$\begin{aligned} E\{h(X_{k,m})\} &= k\binom{m}{k} \int h(u)\{1 - G(u)\}^i\{1 - G(u)\}^{k-i-1} G^{m-k}(u)\, dF(u) \\ &= k\binom{m}{k} \sum_{s=0}^{i} (-1)^s \binom{i}{s} \int h(u) F^{k-i-1}(u)\{1 - F(u)\}^{m-k+s}\, dF(u) \\ &= \binom{m}{k} \sum_{s=0}^{i} (-1)^s (k/(k-i))\left[\binom{i}{s} \Big/ \binom{m-i+s}{k-i}\right] E\{h(X_{k-i,m-i+s})\}. \end{aligned} \tag{2.3}$$

Also, for $1 \leqq k \leqq m \leqq n$ and $0 \leqq j \leqq m-k$,

$$\begin{aligned} E\{h(X_{k,m})\} &= k\binom{m}{k} \int h(u) F^{k-1}(u)\{1 - F(u)\}^j\{1 - F(u)\}^{m-k-j}\, dF(u) \\ &= k\binom{m}{k} \sum_{s=0}^{j} (-1)^s \binom{j}{s} \int h(u) F^{k+s-1}(u)\{1 - F(u)\}^{m-k-j}\, dF(u) \\ &= \binom{m}{k} \sum_{s=0}^{j} (-1)^s (k/(k+s))\left[\binom{j}{s} \Big/ \binom{m-j+s}{k+s}\right] E\{h(X_{k+s,m-j+s})\}. \end{aligned} \tag{2.4}$$

It should be pointed out that formulae (5) and (6) of Srikantan [2] are equivalent respectively to (2.1) with $i = k-1$ and (2.2) with $j = m-k$. The recurrence relations between the moments, between the moment generating functions and between the characteristic functions of the order statistics can be obtained

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from (2.1) and (2.2) by setting $h(u) = u^r$ $(r = 1, 2, \cdots)$, $h(u) = \exp\{tu\}$, (t real) and $h(u) = \exp\{(-1)^{\frac{1}{2}}tu\}$, ($t$ real) respectively. Similarly by letting $h(u) = 1$ if $u \varepsilon (-\infty, x)$ and $h(u) = 0$ otherwise, (2.1) and (2.2) yield recurrence relations between cdf's of order statistics. These in turn lead to results connecting pdf's (provided densities exist). Srikantan [2] makes some similar remarks for his results. He also discusses the interrelation between the results like (2.1) and (2.2) and those available in the literature. For the problem of gamma order statistics, (2.2) was proved in [1] when $h(u) = u^r$.

REFERENCES

[1] GUPTA, S. S. (1960). Order statistics from the gamma distribution. *Technometrics* **2** 243–262.

[2] SRIKANTAN, K. S. (1962). Recurrence relations between the pdf's of order statistics, and some applications. *Ann. Math. Statist.* **33** 169–177.