A NOTE ON INVARIANT MEASURES[1]

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1. Introduction. We consider a Markov process $X_0, X_1, \cdots$ with stationary transition probability function $P(\cdot, \cdot)$ on the state space $(X, \mathbf{B})$, where X is an abstract space and $\mathbf{B}$ a countably generated Borel field of subsets of X. $P^n(\cdot, \cdot)$ denotes the nth iterate of the transition probability function and $P^0(\cdot, E)$ simply means the characteristic function of the set E.

Harris [4] introduced a recurrence condition and proved the existence of an invariant measure under such a condition. Various attempts have been made, see for instance [2], [3], and [5], to replace Harris' condition by a weaker one. The condition imposed by Isaac [5] is apparently weaker as remarked there and in [3]. The main purpose of this note is to show that Isaac's condition [5] is weaker than Harris' [4] only in a trivial sense. This is done in Section 2. This realization seems to give more insight into the results of [3] and [5]. In Section 3 we give another condition equivalent to Isaac's which seems still weaker. Some of the results of [3] are derived as consequences of these observations in Section 4.

We include some definitions and notations in this section. Most of these can be found in [1]. For any E in $\mathbf{B}$ we define

$$L(x, E) = \text{Prob}\,\{X_n \,\varepsilon\, E \quad \text{for some} \quad n \mid X_0 = x\},$$

$$Q(x, E) = \text{Prob}\,\{X_n \,\varepsilon\, E \quad \text{infinitely often} \mid X_0 = x\}.$$

The following relation can easily be verified:

$$Q(x, E) = L(x, E) - \textstyle\sum_{n=1}^{\infty} \int_E P^n(x, dy)[1 - L(y, E)]. \tag{1.1}$$

Definition 1. A nonempty set E in $\mathbf{B}$ is stochastically closed if

$$P(x, E) = 1 \quad \text{for all} \quad x \,\varepsilon\, E.$$

Definition 2. For any E in $\mathbf{B}$,

$$E^\infty = \{x : Q(x, E) = 1\}.$$

The set E^∞ is either empty or stochastically closed by Proposition 4 [1]. The following definition was introduced in [5]:

Definition 3. Let m be a σ-finite measure on $(X, \mathbf{B})$. The process is m-singular if for each x, except for an m-null set, there exists a set L_x, $m(L_x) = 0$, such that $P^n(x, L_x) = 1$ for all positive integers n. In the contrary case the process is called m-non-singular.

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2. Implication of Isaac's condition. We give below three conditions:

CONDITION (C_0) (Harris [4]). There is a σ-finite measure m on $(X, \mathbf{B})$, $m(X) > 0$, such that $m(E) > 0$ implies $Q(x, E) = 1$ for all $x \varepsilon X$.

CONDITION (C_1) (Isaac [5]). There is a σ-finite measure m with respect to which the process is m-non-singular such that $m(E) > 0$ implies $Q(x, E) = 1$ a.e. $(m)x$.

The exceptional set of Condition (C_1) may depend on the set E.

CONDITION (C_2). Everything else is the same as in Condition (C_1) except that the exceptional set is fixed for all E and its complement with respect to X is stochastically closed.

THEOREM 2.1. *Condition* (C_1) *is equivalent to Condition* (C_2).

PROOF. It is enough to show that (C_1) implies (C_2). Hence assume that (C_1) holds. We give first a lemma.

LEMMA 2.1. *Under Condition* (C_1) *there exists a set* C *with* $m(C) > 0$ *such that for some positive integer* n *we have*

$$\inf_{x \varepsilon C, y \varepsilon C} f^n(x, y) \geqq \delta > 0, \tag{2.1}$$

where $f^n(x, \cdot)$ *is the density of the absolutely continuous part of* $P^n(x, \cdot)$ *with respect to* m.

PROOF. Lemma 2[5] implies that if r is any real number, $0 < r < 1$, there exist a set $B \varepsilon \mathbf{B}$ and a positive integer k such that

$$0 < m(B) < \infty, \qquad \text{and for every } x \varepsilon B:$$

$$m\{y : y \varepsilon B, f^1(x, y) + \cdots + f^k(x, y) > k^{-1}\} > rm(B). \tag{2.2}$$

Moreover $Q(x, B) = 1$ for all $x \varepsilon B$. The rest of the argument is same as the proofs of Lemma 2.1 [7] and Theorem 2.1 [7] since (2.2) is all that is needed to carry forth the argument. For a minor correction to the proof of Lemma 2.1 [7] we refer to Section 4 [6].

We now complete the proof of the theorem. Let $E \subset C$ with $m(E) > 0$. Then (2.1) implies

$$\inf_{x \varepsilon C} L(x, E) \geqq \delta m(E) > 0. \tag{2.3}$$

It follows from Proposition 7 [1] that for all $x \varepsilon X$, $Q(x, C) \leqq Q(x, E)$. In particular $Q(x, E) = 1$ for all $x \varepsilon C^\infty$. Let now E be any set in $\mathbf{B}$ with $m(E) > 0$. By Condition (C_1) there is a set F contained in C with $m(F) > 0$, and such that for all $x \varepsilon F$, $Q(x, E) = 1$. Applying Proposition 7 [1] again we have $Q(x, F) \leqq Q(x, E)$ for all $x \varepsilon X$. Since $F \subset C$, it follows from above that $Q(x, F) = 1$ for all $x \varepsilon C^\infty$ and hence $Q(x, E) = 1$ for all $x \varepsilon C^\infty$. C^∞ is stochastically closed since it is nonempty and its complement, which becomes the exceptional set of Condition (C_2), must be m-null as a consequence of Condition (C_1). This completes the proof of the theorem.

3. Another condition equivalent to Condition (C_1). The following condition will be proved equivalent to (C_1).

CONDITION (C_3). There exists a σ-finite measure m satisfying the following:

(i) $m(E) = 0$ implies $P^n(x, E) = 0$ a.e. $(m)x$, for all $n \geqq 0$,

(ii) the process is m-non-singular and $m(E) > 0$ implies $\sum_{n=0}^{\infty} P^n(x, E) = \infty$ a.e. $(m)x$. The exceptional set is not assumed fixed.

THEOREM 3.1. *Conditions* (C_1) *and* (C_3) *are equivalent.*

PROOF. We first show that (i) of Condition (C_3) can be assumed to hold for the measure m of (C_1) without any loss of generality. Let m be the measure of (C_1) and consider the measure $\tilde{m}$ given by $\tilde{m} = \sum_{n=0}^{\infty} 2^{-n} T^n m$ where

$$T^n m = \int P^n(x, \cdot) m(dx).$$

We claim that $\tilde{m}$ can replace m in (C_1) and it also satisfies (i) of (C_3). For the first assertion it is enough to show that Tm can replace m in (C_1). The definition of T implies that if $P^n(x, \cdot)$ has non-trivial absolutely continuous part with respect to m then $P^{n+1}(x, \cdot)$ has non-trivial absolutely continuous part with respect to Tm. Hence if the process is m-non-singular it is Tm-non-singular. Next thing to show is that $Tm(E) > 0$ implies $Q(x, E) = 1$ a.e. (Tm). $Tm(E) > 0$ implies $P(y, E) > \epsilon > 0$ for $y \varepsilon$ some set E_0, $m(E_0) > 0$. Hence $\inf_{y \varepsilon E_0} L(y, E) \geqq \epsilon > 0$. By Proposition 7 [1] it follows that for all $x \varepsilon X$, $Q(x, E_0) \leqq Q(x, E)$. Since $Q(x, E_0) = 1$ a.e. (m) we conclude that $Q(x, E) = 1$ a.e. (m). Let $F = X - E^{\infty}$. We have to show that $Tm(F) = 0$. Suppose $Tm(F) > 0$, then following the reasoning above we conclude that $Q(x, F) = 1$ a.e. $(m)x$ which is a contradiction because we never hit F from E^{∞} and $m(E^{\infty}) = m(X)$. Thus $Tm(F) = 0$. That $\tilde{m}$ satisfies (i) of (C_3) is obvious.

To complete the proof it now suffices to show that (C_3) implies (C_1). Assume (C_3). For $\delta > 0$, let $E_\delta = \{x \varepsilon E : L(x, E) < 1 - \delta\}$. It follows from (1.1) that for all $x \varepsilon X$, $\sum_{n=1}^{\infty} P^n(x, E_\delta) < \delta^{-1} < \infty$, and since ($C_3$) is assumed we must have $m(E_\delta) = 0$. Consequently $L(x, E) = 1$ a.e. (m) on E. Using (1.1) again together with (i) of (C_3) we conclude that $Q(x, E) = 1$ a.e. (m) on E. Thus E^{∞} is not empty, hence stochastically closed. Thus $m(X - E^{\infty}) = 0$ and $Q(x, E) = 1$ a.e. $(m)x$. The theorem is proved.

4. Equivalence of the invariant measure to $\tilde{m}$ and its uniqueness under Isaac's condition. We can replace m by $\tilde{m}$ in Isaac's condition as shown in the proof of Theorem 3.1. Referring back to Section 2 we pick the set C corresponding to $\tilde{m}$ and consider the process on C^{∞}. This process satisfies (C_0) with the measure $\tilde{m}$. Harris' Theorem 1 [4] implies the existence of a unique invariant measure which is stronger than $\tilde{m}$ for the C^{∞}-process. Let π denote this measure. Define $\pi(X - C^{\infty}) = 0$. Since $\tilde{m}(X - C^{\infty}) = 0$ we still have π stronger than $\tilde{m}$. We show that π is actually equivalent to $\tilde{m}$. Suppose E is an $\tilde{m}$-null set. Then $P^n(x, E) = 0$ a.e. $(\tilde{m})x$ for all $n \geqq 0$. On the other hand, if $\pi(E) > 0$, then $Q(x, E) = 1$ for all $x \varepsilon C^{\infty}$ as a consequence of a remark in Harris [4], this is a contradiction. Hence $\tilde{m}(E) = 0$ implies $\pi(E) = 0$ and π is indeed equivalent to $\tilde{m}$. That π is unique among invariant measures equivalent to $\tilde{m}$ is also clear from Theorem 2.1 and Theorem 1 [4].

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