REMARK ON THE OPTIMUM CHARACTER OF THE SEQUENTIAL PROBABILITY RATIO TEST[1]

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There is a small lacuna in the proof ([1]) of the property stated in the title of this note. In some recent papers many pages are devoted to correcting it. Since all "other" proofs of the optimum character of the sequential probability ratio test follow all the principal ideas of [1] and differ from the latter only in very minor details, it seems appropriate to show, as we will, that the lacuna can be filled in a very simple and obvious way. The present note assumes familiarity only with Lemmas 1, 2, and 3 of [1]. The gap in [1] is in Lemma 1, where it is claimed that the test S^* there constructed minimizes the average risk.

We shall replace Lemmas 1, 2, and 3 of [1] by Lemma A whose statement is that of Lemma 1 plus that of Lemma 2 plus that of Lemma 3. The proof of Lemma A will be that given for Lemma 1, followed by that given for Lemma 2, followed by that given for Lemma 3, followed by the remarks which we now make.

At the end of the proof of Lemma 2 we already have that S^* is the sequential probability ratio test.

We now prove that S^* minimizes the average risk. We adopt the notation and terminology of [1]. Suppose there were a test S such that

$$R(S) = R(S^*) - \delta, \qquad \delta > 0. \tag{1}$$

We shall construct a sequence of tests $S_0(=S)$, S_1, S_2, $\cdots$, such that, for $i = 0, 1, 2, \cdots$,

$$R(S_{i+1}) \leqq R(S_i) + \delta(2^{-i-2}) \tag{2}$$

and

$$\lim_{i\to\infty} R(S_i) = R(S^*). \tag{3}$$

From this it follows that

$$R(S^*) \leqq R(S) + \delta/2. \tag{4}$$

The contradiction between (1) and (4) proves the desired result.

If t is any sequential test let $n(t)$ be its associated stopping variable; the value of $n(t)$ at the point $\omega = x_1, x_2, \cdots$ will be denoted by $n(\omega, t)$. Let $r_j(\omega) = r_j(x_1, \cdots, x_j) = p_{1j}/p_{0j}$ as in [1]. Let T be the totality of all sequential tests t such that $E_i[n(t)] < \infty$, $i = 0, 1$. Define

$$T_0 = \{t \,\varepsilon\, T \mid r_j(\omega) \geqq A \quad \text{or} \quad \leqq B \Rightarrow n(\omega, t) \leqq j, j \geqq 1\}$$

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and

$$T_j = \{t \,\varepsilon\, T_0 \mid P_i\{\omega \mid n(\omega, t) = j, B < r_j(\omega) < A\} = 0, i = 0, 1\},$$

where P_i indicates probability under the hypothesis H_i. We note that, if S is not in T_0, we may, by Lemma 3, (c), replace it by an obvious modification $S' \,\varepsilon\, T_0$ such that $R(S') \leqq R(S)$. We therefore assume that S is in T_0. Let α be the integer such that $S \,\varepsilon\, \bar{T}_\alpha \bigcap_{i=1}^{\alpha-1} T_i$.

Let $\epsilon > 0$ be arbitrary. It follows from Lemma 3 and the argument of Lemma 2 that there exists a sequential test $Z(\epsilon)$ (resp. $Z'(\epsilon)$) which requires at least one additional observation, and is such that

$$l(x_1, \cdots, x_\alpha) - E(L_n \mid (x_1, \cdots, x_\alpha), Z(\epsilon)) > -\epsilon$$

for any $(x_1, \cdots, x_\alpha)$ such that $B \leqq r_\alpha(x_1, \cdots, x_\alpha) \leqq W_0 g_0/W_1 g_1$ (resp., is such that

$$l(x_1, \cdots, x_\alpha) - E(L_n \mid (x_1, \cdots, x_\alpha), Z'(\epsilon)) > -\epsilon$$

for any $(x_1, \cdots, x_\alpha)$ such that $W_0 g_0/W_1 g_1 < r_\alpha(x_1, \cdots, x_\alpha) \leqq A$.) Now modify S as follows: when $B < r_\alpha(x_1, \cdots, x_\alpha) \leqq W_0 g_0/W_1 g_1$ (resp., when $W_0 g_0/W_1 g_1 < r_\alpha(x_1, \cdots, x_\alpha) < A$) and S calls for stopping taking observations, replace S by the test $Z(\delta/4)$ (resp., $Z'(\delta/4)$). Call the resulting test S_1. Since S, Z, and Z' fulfill the required measurability and integrability conditions, so does S_1, which also satisfies (2) for $i = 0$.

We repeat the above procedure in an obvious way on S_1, replacing $\delta/4$ by $\delta/8$, and obtaining S_2. We repeat the same procedure on $S_2, S_3, \cdots$, using $\delta/16, \delta/32, \cdots$. Since the index α increases by at least one each time, (3) follows. this completes the proof.

REFERENCE

[1] Wald, A. and Wolfowitz, J. (1948). Optimum character of the sequential probability ratio test. *Ann. Math. Statist.* **19** 326–339.