ON A THEOREM OF CRAMÉR AND LEADBETTER[1]

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1. Introduction. In a recent paper [1], Cramér and Leadbetter have given an integral formula for the kth factorial moment of the number of upcrossings of the zero level by a stationary Gaussian process in unit time. More specifically, suppose $X(\cdot)$ is such a process and that $X'(\cdot)$ exists and has continuous sample paths. If N is the number of upcrossings of zero by $X(\cdot)$, there is a positive function f defined on the unit cube Ω in k-space and dependent on the joint densities of $X(\cdot)$ and $X'(\cdot)$ for which $EN(N-1)\cdots(N-k+1) = \int_\Omega f\,d\mu$, μ Lebesgue measure. From this result, one may also give expression to the kth factorial moment of the number of zeros of $X(\cdot)$.

A second fact is established in [1], viz., $\int_\Omega f\,d\mu \leqq EN(N-1)\cdots(N-k+1)$ even if $X'(\cdot)$ has discontinuous sample paths. Consequently, the formula still holds provided f is not integrable. At present, the relationship between (a) f integrable and (b) $X'(\cdot)$ has continuous sample paths is not known.

In this paper we find for quite general processes, a particular submartingale (relative to the Lebesgue measure space) sequence $\{f_n\}$ of functions on Ω for which $\int_\Omega \liminf f_n\,d\mu \leqq EN(N-1)\cdots(N-k+1)$. Under suitable conditions, $f_n \to_{\text{a.s.}} f$ and $\int_\Omega f\,d\mu = EN(N-1)\cdots(N-k+1)$. As a special case, this is shown to hold for the processes of [1] without the continuity restriction on $X'(\cdot)$ (to achieve this, $X(\cdot)$ is subjected to a nondegeneracy requirement also required in [1]).

2. Moments of upcrossings. Let $X(\cdot)$ be a separable stochastic process on the unit interval. We assume throughout that $X(t)$ has a continuous distribution for each $t \in [0, 1]$ and that $X(\cdot)$ has continuous sample paths with probability 1.

The number N of upcrossings of zero by the process $X(\cdot)$ is approximated by counting those of a polygonal process tied to $X(\cdot)$ at points of the form $i/2^n$. Specifically, let U_{ni} be the indicator of the event $\{X[(i-1)/2^n] < 0 < X(i/2^n)\}$. Under the above assumptions, $\sum' U_{ni_1}\cdots U_{ni_k} \uparrow N(N-1)\cdots(N-k+1)$ a.s., where $\sum'$ denotes summation over all appropriate $\mathbf{i} = (i_1, \cdots, i_k)$ having distinct entries (cf [1]). Then, according to the monotone convergence theorem,

$$(1)\quad EN(N-1)\cdots(N-k+1) = \lim_{n\to\infty} \sum{}' P\{X[(i_1-1)/2^n] < 0 < X(i_1/2^n), \cdots, X[(i_k-1)/2^n] < 0 < X(i_k/2^n)\}.$$

We will subsequently consider as an auxiliary probability space the unit cube Ω in k-space, together with the Lebesgue measurable subsets and Lebesgue meas-

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ure μ. In this context, let f_n be a function defined a.s. on Ω by

$$
\begin{aligned}
f_n(\mathbf{t}) &= 2^{kn}P\{X[(i_1-1)/2^n] < 0 < X(i_1/2^n), \cdots, X[(i_k-1)/2^n] \\
&\qquad < 0 < X(i_k/2^n)\} \quad \text{if} \quad \mathbf{t} = (t_1, \cdots, t_k) \quad \text{and} \\
&\qquad (i_j-1)/2^n < t_j < i_j/2^n, \qquad i_j \text{ distinct}, \ j = 1, \cdots, k, \qquad (2)\\
f_n(\mathbf{t}) &= 0 \quad \text{if} \quad \mathbf{t} = (t_1, \cdots, t_k) \quad \text{and} \quad (i_j-1)/2^n < t_j < i_j/2^n, \\
&\qquad j = 1, \cdots, k, i_j \text{ not all distinct.}
\end{aligned}
$$

In particular, $f_n(\mathbf{t}) = 0$ if $i_j = i_m + 1$ for some j and m. (1) and (2) together imply

$$
\begin{aligned}
&\textstyle\sum' P\{X[(i_1-1)/2^n] \\
&\qquad < 0 < X(i_1/2^n), \cdots, X[(i_k-1)/2^n] < 0 < X(i_k/2^n)\} \qquad (3)\\
&\qquad\qquad = \textstyle\int_\Omega f_n\, d\mu \uparrow EN(N-1) \cdots (N-k+1).
\end{aligned}
$$

Furthermore, if f denotes $\liminf f_n$, we have

$$\textstyle\int_\Omega f\, d\mu \leqq EN(N-1) \cdots (N-k+1). \qquad (4)$$

By making suitable assumptions, f will be the a.s. limit of the f_n. Indeed, it will be seen that $\{f_n\}$ forms a submartingale and the a.s. existence of the limit will follow from the martingale convergence theorem if $EN(N-1) \cdots (N-k+1)$ is assumed finite. More to the point, a dominated assumption placed on the f_n will ensure that $f_n \to_{\text{a.s.}} f$ and that equality holds in (4) whether $EN(N-1) \cdots (N-k+1)$ is finite or not.

Lemma. *Let* $C = \{\mathbf{t} = (t_1, \cdots, t_k) \mid (i_j-1)/2^n < t_j < i_j/2^n, j = 1, 2, \cdots, k\}$ *for some choice of* $i_1, \cdots, i_k$, *then* $\int_C f_n\, d\mu \leqq \int_C f_{n+1}\, d\mu$.

Proof. From the definition of f_n and the remark immediately below it, the only case to be checked has $|i_j - i_m| > 1$ for $j \neq m$. Clearly,

$$
\begin{aligned}
\textstyle\int_C f_n\, d\mu &= P\{X[(i_1-1)/2^n] \\
&\qquad < 0 < X(i_1/2^n), \cdots, X[(i_k-1)/2^n] < 0 < X(i_k/2^n)\} \\
&\qquad\qquad = P(U_{ni_1} \cdots U_{ni_k} = 1).
\end{aligned}
$$

Returning to the definition of U_{ni}, it can be seen generally that $U_{ni} = 1$ implies $U_{n+1,2i-1} + U_{n+1,2i} = 1$ a.s. and that $U_{n+1,2i-1} \cdot U_{n+1,2i} = 0$. Therefore,

$$
\begin{aligned}
P[U_{ni_1} \cdots U_{ni_k} = 1] &\leqq P[(U_{n+1,2i_1-1} + U_{n+1,2i_1}) \cdots (U_{n+1,2i_k-1} + U_{n+1,2i_k}) = 1] \\
&= P[\textstyle\sum^* U_{n+1,r_1} \cdots U_{n+1,r_k} = 1] \\
&= \textstyle\sum^* P[U_{n+1,r_1} \cdots U_{n+1,r_k} = 1]
\end{aligned}
$$

where $\sum^*$ denotes summation over all terms obtained by setting $r_j = 2i_j - 1$ or $r_j = 2i_j$, $j = 1, \cdots, k$. Now $P[U_{n+1,r_1} \cdots U_{n+1,r_k} = 1] = \int_{C_r} f_{n+1}\, d\mu$ with

$C_r = \{\mathbf{t} = (t_1, \cdots, t_k) \mid (r_j - 1)/2^{n+1} < t_j < r_j/2^{n+1}, j = 1, \cdots, k\}$ so $\sum^* P[U_{n+1,r_1} \cdots U_{n+1,r_k} = 1] = \sum^* \int_{C_r} f_{n+1}\, d\mu = \int_{U^* C_r} f_{n+1}\, d\mu = \int_C f_{n+1}\, d\mu$ and the lemma is proved.

We have established then that $\{f_n\}$ forms a submartingale. To account for those situations in which $EN(N-1) \cdots (N-k+1)$ is infinite and equality still holds in (4), the following appears to be most useful.

THEOREM. *If for each $\epsilon > 0$, $f_n \leqq g_\epsilon$ on $A_\epsilon = \{\mathbf{t} = (t_1, \cdots, t_k) \mid |t_i - t_j| \geqq \epsilon, i \neq j\}$ with g_ϵ integrable, then $f_n \rightarrow_{a.s.} f$ and $EN(N-1) \cdots (N-k+1) = \int_\Omega f\, d\mu$, finite or not.*

PROOF. Let $B_n = \{\mathbf{t} = (t_1, \cdots, t_k) \mid (i_j - 1)/2^n < t_j < i_j/2^n$ for some $\mathbf{i} = (i_1, \cdots, i_k), |i_j - i_m| > 1, j \neq m\}$. The sequence $\{f_{n+r} I_{B_n}\}$ is a submartingale sequence in $r \geqq 0$ for each fixed n and $I_{B_n} \uparrow 1$ a.s. as $n \rightarrow \infty$. For each fixed n, the sequence $\{f_{n+r} I_{B_n}\}$ is dominated by an integrable function ($B_n \subset A_{2^{-n}}$) and hence there is a function f on Ω for which $f_n \rightarrow_{a.s.} f$ and $\int_{B_n} f_{n+r}\, d\mu \rightarrow \int_{B_n} f\, d\mu$ as $r \rightarrow \infty$. Using the submartingale property,

$$\int_\Omega f_n\, d\mu = \int_{B_n} f_n\, d\mu \leqq \int_{B_n} f_{n+r}\, d\mu \rightarrow \int_{B_n} f\, d\mu \leqq \int_\Omega f\, d\mu$$

so that

$$\int_\Omega f\, d\mu \geqq \lim_{n\to\infty} \int_\Omega f_n\, d\mu = EN(N-1) \cdots (N-k+1)$$

and the conclusion follows.

Before turning to normal processes, suppose for the moment that the random variables $X[(i_1 - 1)/2^n], \cdots, X[(i_k - 1)/2^n], 2^n\{X(i_1/2^n) - X[(i_1 - 1)/2^n]\}, \cdots, 2^n\{X(i_k/2^n) - X[(i_k - 1)/2^n]\}$ have a $2k$-dimensional continuous density function for each choice of $n, i_1, \cdots, i_k, |i_j - i_m| > 1, j \neq m$. If $p_{n,\mathbf{t}}$ denotes this density for $\mathbf{t} = (t_1, \cdots, t_k)$, $(i_j - 1)/2^n < t_j < i_j/2^n, j = 1, \cdots, k$, then (cf [1]),

$$\begin{aligned} f_n(\mathbf{t}) &= 2^{kn} \int_0^\infty \cdots \int_0^\infty dy_1 \cdots dy_k \int_{-2^{-n}y_1}^0 \cdots \int_{-2^{-n}y_k}^0 \\ &\qquad \cdot p_{n,\mathbf{t}}(x_1, \cdots, x_k, y_1, \cdots, y_k)\, dx_1 \cdots dx_k \qquad (5) \\ &= \int_0^\infty \cdots \int_0^\infty y_1 \cdots y_k p_{n,\mathbf{t}}(\theta_{1n}, \cdots, \theta_{kn}, y_1, \cdots, y_k)\, dy_1 \cdots dy_k, \\ &\qquad -2^{-n} y_j < \theta_{jn} < 0, j = 1, \cdots, k. \end{aligned}$$

3. Gaussian processes. Let $X(\cdot)$ be a stationary Gaussian process with mean function zero and a spectral distribution function having an absolutely continuous part so that (5) holds. It is assumed that $X(\cdot)$ has a quadratic mean derivative $X'(\cdot)$ (and there is no loss of generality in assuming this, for otherwise $EN = +\infty$, [3]).

For fixed $\mathbf{t} \varepsilon B_n$, write $\Sigma_{n,\mathbf{t}}$ for the covariance matrix of $X[(i_1 - 1)/2^n], \cdots, X[(i_k - 1)/2^n], 2^n\{X(i_1/2^n) - X[i_1 - 1/2^n]\}, \cdots, 2^n\{X(i_k/2^n) - X[(i_k - 1)/2^n]\}$, $i_1, \cdots, i_k$ as in (2). As $n \rightarrow \infty$, $\Sigma_{n,\mathbf{t}} \rightarrow \Sigma_\mathbf{t}$, the covariance matrix of $X(t_1), \cdots, X(t_k), X'(t_1), \cdots, X'(t_k)$, also nonsingular. Pointwise, the integrand in the last integral of (5) converges to $y_1 \cdots y_k p_\mathbf{t}\, (0, \cdots, 0, y_1,$

$\cdots, y_k)$ where $p_{\mathbf{t}}$ is the $2k$-dimensional normal density with mean zero and covariance $\Sigma_{\mathbf{t}}$. Moreover, the sequence of integrands may be bounded above by an integrable function of the form $C \cdot y_1 \cdots y_k e^{-\frac{1}{2}\mathbf{y}' B \mathbf{y}}$. Thus,

$$(6) \quad f_n(\mathbf{t}) \to f(\mathbf{t}) = \int_0^\infty \cdots \int_0^\infty y_1 \cdots y_k p_{\mathbf{t}}(0, \cdots, 0, y_1, \cdots, y_k) \, dy_1 \cdots dy_k .$$

(6) remains true for $\mathbf{t}$ having coordinates of the form $m/2^r$ when $f_n(\mathbf{t})$, $n \geqq r$, is given any value which is a limit point of $f_n(\mathbf{s})$ as $\mathbf{s} \to \mathbf{t}$ i.e., the corresponding i may be chosen to be either $2^{n-r} \cdot m$ or $2^{n-r}m - 1$.

The function f of (6) is continuous over each A_ϵ and hence is bounded there. This in turn implies that the f_n are uniformly bounded over each A_ϵ. To see this, note that if $f_n \geqq M$ on some $C_n = \{\mathbf{t} \mid (i_j - 1)/2^n < t_j < i_j/2^n, j = 1, \cdots, k\}$, then $f_{n+1} \geqq M$ on some $C_{n+1} \subset C_n$ of the same form. If f_{n+m} is extended to be continuous over the closure $\bar{C}_{n+m}$ of C_{n+m}, there is a point $\mathbf{t}_0 = \bigcap_{m \geqq 0} \bar{C}_{n+m}$ for which $f_{n+m}(\mathbf{t}_0) \geqq M$ for all $m \geqq 0$ and so $f(\mathbf{t}_0) \geqq M$. Applying the theorem of the previous section, the result of [1] is obtained under relaxed assumptions.

There is no essential use made of stationarity in the preceding argument and a corresponding formula will hold for nonstationary Gaussian processes as long as $X'(t)$ exists in quadratic mean at each $t \in [0, 1]$ and the covariance matrices $\Sigma_{n,\mathbf{t}}$ and $\Sigma_{\mathbf{t}}$ are nonsingular for $\mathbf{t} = (t_1, \cdots, t_k)$ having distinct coordinates.

Finally, Leadbetter has pointed out in [2] that, for Gaussian processes which are nondegenerate in the above sense, it is possible to reconvert the factorial moments of the number of upcrossings to the factorial moments of the number of zeros.

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