ADMISSIBILITY OF CONFIDENCE INTERVALS

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1. Introduction. Hodges and Lehmann (1951) have shown that for a sample of n independent observations from a univariate normal population the sample mean is an admissible estimator of the parent mean. More general results have recently been proved by Farrell (1964) regarding the admissibility of estimators of the location parameter in a class of continuous frequency functions. The analogous question regarding confidence intervals is considered here, and the admissibility of a class of confidence intervals is proved for the location parameter in a wide class of continuous frequency functions which includes the normal and some other commonly occurring ones. A practically important application of the result is that the usual symmetrical confidence intervals for the mean of a normal population based on the 't' statistic are seen to be admissible whether the population variance is known or not.

Again, the general result which is proved for confidence intervals whose length may be any random variable distributed independently of the location parameter under estimation, also includes as a particular case the admissibility of certain well known confidence intervals of constant length, obtained by minimizing the length for a given confidence level. For this particular case, however, the result can be established under less restrictive assumptions, either by a direct proof or as a deduction from Farrell's results (1964).

2. Notation. In the following, X denotes a real random variable with a df involving a parameter θ which assumes values in a set Ω of the real line; $x_1, \cdots, x_n$ independent observations of X; and $x = (x_1, x_2, \cdots, x_n)$ a point in the sample space $\mathfrak{X}$; on $\mathfrak{X}$ and Ω is defined the Lebesgue measure, all sets being Lebesgue measurable; $a(x)$, $b(x)$, with or without subscripts denote measurable functions defined on $\mathfrak{X}$, and $(a(x), b(x))$ denotes the set of confidence intervals $[a(x) \leqq \theta \leqq b(x)]$. We define admissibility of confidence intervals as below:

Definition 2.1. A set of confidence intervals $(a(x), b(x))$ is said to be admissible if and only if, there exists no other set of confidence intervals $(a_1(x), b_1(x))$ satisfying

(i) $b_1(x) - a_1(x) \leqq b(x) - a(x)$ for almost all $x \in \mathfrak{X}$, and

(ii) $P(a_1(x) \leqq \theta \leqq b_1(x) \mid \theta) \geqq P(a(x) \leqq \theta \leqq b(x) \mid \theta)$ for all $\theta \in \Omega$, the strict inequality in (ii) holding for at least one $\theta \in \Omega$. The definition of admissibility for confidence intervals was formulated by Godambe (1961) but in his formulation, the strict inequality in (ii) was required to hold on a non-null set of Ω. We have slightly modified his definition to make it agree with the conventional concept of

Received 4 January 1965.

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admissibility. Godambe has also defined in his paper (1961) related Bayes shortest confidence regions which have later been applied to χ^2-estimation by Box and Tiao (1965) and Watson (1965). But the confidence intervals in the present paper are not Bayes intervals.

3. Main result. We now prove the following

THEOREM 3.1. *If $x_1, x_2, \cdots, x_n$ are independent observations from a known frequency function $f(x, \theta)$ containing an unknown parameter θ, $-\infty < \theta < +\infty$, and if,*

(a) *$f(x, \theta)$ admits a sufficient statistic $\bar{x}$ for θ, where $\bar{x}$ is a function of $x_1, x_2, \cdots, x_n$ with a frequency function of the form $p(\bar{x} - \theta)$* i.e. *the distribution of $(\bar{x} - \theta)$ given θ is independent of θ for $-\infty < \theta < +\infty$,*

(b) *the frequency function $p(t)$ in* (a) *strictly decreases for $t \geqq 0$ as t increases, and for $t \leqq 0$ as t decreases; is continuous for all t and is such that*

$$\int_0^\infty dt_1 [\int_{t_1}^\infty p(t)\, dt + \int_{-\infty}^{-t_1} p(t)\, dt]$$

converges,

(c) *the frequency function $f(x, \theta)$ of x is positive (> 0) and continuous in x for all $x = (x_1, x_2, \cdots, x_n)$ and all θ, $-\infty < \theta + \infty$,*

(d) *$v_1(x)$, $v_2(x)$ are non-negative statistics distributed independently of $\bar{x}$ and θ such that for every $x \varepsilon \mathfrak{X}$ and $p(t)$ in* (a), *$p(-v_2(x)) = p(v_1(x))$; and further such that $v(x) = \max(v_1(x), v_2(x))$ has finite expectation and variance; then, the confidence interval for θ: $[\bar{x} - v_1(x) \leqq \theta \leqq \bar{x} + v_2(x)]$ are admissible according to the Definition* 2.1.

PROOF. If the confidence intervals specified in the theorem are not admissible then by Definition 2.1, there exist functions $u_1(x)$, $u_2(x)$ such that

$$u_2(x) - u_1(x) \leqq v_1(x) + v_2(x) \quad \text{a.e. in} \quad \mathfrak{X} \tag{1}$$

and

$$P[u_1(x) \leqq \theta \leqq u_2(x) \mid \theta] \geqq P[\bar{x} - v_1(x) \leqq \theta \leqq \bar{x} + v_2(x) \mid \theta] \tag{2}$$

for all θ, $-\infty < \theta < +\infty$, the strict inequality holding for at least one θ.

Now put,

$$u(x) = [v_1(x)u_2(x) + v_2(x)u_1(x)]/[v_1(x) + v_2(x)] \tag{3}$$

provided $v_1(x)$ and $v_2(x)$ do not both vanish. If $v_1(x) = v_2(x) = 0$, then noting (1), we put

$$u(x) = u_1(x) = u_2(x). \tag{4}$$

Now noting (1), $u(x) - u_1(x) = v_1(x)[u_2(x) - u_1(x)]/[v_1(x) + v_2(x)] \leqq v_1(x)$, so that, $u(x) - v_1(x) \leqq u_1(x)$; Similarly,

$$u_2(x) - u(x) = v_2(x)[u_2(x) - u_1(x)]/[v_1(x) + v_2(x)] \leqq v_2(x),$$

so that, $u(x) + v_2(x) \geqq u_2(x)$; hence from (2),

$$P[u(x) - v_1(x) \leqq \theta \leqq u(x) + v_2(x) \mid \theta] \geqq P[\bar{x} - v_1(x) \leqq \theta \leqq \bar{x} + v_2(x) \mid \theta] \tag{5}$$

for all θ, $-\infty < \theta < +\infty$, the strict inequality holding for at least one θ.

We next define for given θ, sets $\subset \mathfrak{X}$ as follows

$$\begin{aligned}
D_\theta &= [x : \bar{x} - v_1(x) \leqq \theta \leqq \bar{x} + v_2(x)], \\
E_\theta &= [x : u(x) - v_1(x) \leqq \theta \leqq u(x) + v_2(x)], \\
A_\theta &= D_\theta - D_\theta \cdot E_\theta \text{ where } D_\theta \cdot E_\theta \text{ denotes the intersection set of } D_\theta \\
&\quad \text{and } E_\theta . \\
B_\theta &= E_\theta - D_\theta \cdot E_\theta , \\
T_a &= [x : |x| \leqq a - v(x)] \quad \text{where } a \text{ is an arbitrary number } > 0 \\
&\qquad \text{and } v(x) = \max(v_1(x), v_2(x)), \text{ and} \\
T_a^c &= [x : |x| > a - v(x)].
\end{aligned} \tag{6}$$

Then from (5) and (6), for all θ, $-\infty < \theta < +\infty$, $P(E_\theta \mid \theta) \geqq P(D_\theta \mid \theta)$ so that $P(B_\theta \mid \theta) \geqq P(A_\theta \mid \theta)$, and hence

$$\begin{aligned}
P(B_\theta \cdot T_a^c \mid \theta) &\geqq P(A_\theta \mid \theta) - P(B_\theta \cdot T_a \mid \theta) \\
&\geqq P(A_\theta \cdot T_a \mid \theta) - P(B_\theta \cdot T_a \mid \theta) \\
&= P(D_\theta \cdot T_a \mid \theta) - P(E_\theta \cdot T_a \mid \theta),
\end{aligned}$$

from which by integrating w.r.t. θ from $-a$ to $+a$, we get,

$$\int_{-a}^{a} P(B_\theta \cdot T_a^c \mid \theta)\, d\theta \geqq \int_{-a}^{a} P(D_\theta \cdot T_a \mid \theta)\, d\theta - \int_{-a}^{a} P(E_\theta \cdot T_a \mid \theta). \tag{7}$$

Since by (a), $\bar{x}$ is sufficient for θ, the frequency function $\prod_{k=1}^{n} f(x_k \cdot \theta) = p(\bar{x} - \theta) \cdot L(x_1, \cdots, x_n \mid \bar{x}) = p(\bar{x} - \theta) \cdot L(x \mid \bar{x})$. Hence in the r.h.s. of (7), $\int_{-a}^{a} P(D_\theta \cdot T_a \mid \theta)\, d\theta = \int_{-a}^{a} d\theta [\int_{D_\theta \cdot T_a} L(x \mid \bar{x}) \cdot p(\bar{x} - \theta)\, dx]$ which by interchanging the order of integration w.r.t. $x = (x_1, x_2, \cdots, x_n)$ and θ, and noting the Definitions (6) of D_θ and T_a, equals

$$\int_{T_a} L(x \mid \bar{x})\, dx \left[\int_{\bar{x} - v_1(x)}^{\bar{x} + v_2(x)} p(\bar{x} - \theta)\, d\theta\right] = \int_{T_a} L(x \mid \bar{x})\, dx \left[\int_{-v_2(x)}^{v_1(x)} p(t)\, dt\right]. \tag{8}$$

Similarly putting,

$$g_2(x) = \min.\,(+a, u(x) + v_2(x)) \quad \text{and} \quad g_1(x) = \max.\,(-a, u(x) - v_1(x))$$

we have the second term in the r.h.s. of (7),

$$\begin{aligned}
\int_{-a}^{a} P(E_\theta \cdot T_a \mid \theta)\, d\theta &= \int_{-a}^{a} d\theta \left[\int_{E_\theta \cdot T_a} L(x \mid \bar{x}) \cdot p(\bar{x} - \theta)\, dx\right] \\
&= \int_{T_a} L(x \mid \bar{x})\, dx \left[\int_{g_1(x)}^{g_2(x)} p(\bar{x} - \theta)\, d\theta\right] \\
&\leqq \int_{T_a} L(x \mid \bar{x})\, dx \left[\int_{u(x) - v_1(x)}^{u(x) + v_2(x)} p(\bar{x} - \theta)\, d\theta\right] \\
&= \int_{T_a} L(x \mid \bar{x})\, dx \left[\int_{\bar{x} - u(x) - v_2(x)}^{\bar{x} - u(x) + v_1(x)} p(t)\, dt\right].
\end{aligned} \tag{9}$$

Substituting (8) and (9),

$$(10)\quad \text{r.h.s. of (7)} \quad \geqq \int_{T_a} L(x \mid \bar{x})\, dx \left[\int_{-v_2(x)}^{v_1(x)} p(t)dt - \int_{-v_2(x)+\bar{x}-u(x)}^{v_1(x)+\bar{x}-u(x)} p(t)\, dt\right].$$

Since by (d), $p(v_1(x)) = p(-v_2(x))$ and by (b), $p(t)$ is strictly decreasing, in the r.h.s. of (10), the factor,

$$(11)\qquad \left[\int_{-v_2(x)}^{v_1(x)} p(t)\, dt - \int_{-v_2(x)+\bar{x}-u(x)}^{v_1(x)+\bar{x}-u(x)} p(t)\, dt\right] \quad \text{is always} \quad \geqq 0.$$

The equality in (11) holding if, and only if, $\bar{x} = u(x)$. Thus the r.h.s. of (10) is always non-negative and as the set T_a increases monotonically as a increases, the r.h.s. of (10) is always non-decreasing as a increases. We next show that it is bounded above.

Let M_a, M_a^c be the sets $\subset \mathfrak{X}$ defined by

$$(12)\qquad M_a = \left[x : v(x) \leqq \frac{a}{2}\right]; \qquad M_a^c = \left[x : v(x) > \frac{a}{2}\right].$$

Since by (d) $E(v(x) \mid \theta) = E(v(x))$ exists and is independent of θ, $P(M_a^c \mid \theta) \leqq 2E(v(x))/a$ and hence,

$$(13)\qquad \int_{-a}^{a} P(M_a^c \mid \theta)\, d\theta \leqq 4E(v(x)).$$

Now, noting (6), in the l.h.s. of (7),

$$P(B_\theta \cdot T_a^c \mid \theta) = P(B_\theta \cdot M_a \cdot T_a^c \mid \theta) + P(B_\theta \cdot M_a^c \cdot T_a^c \mid \theta)$$
$$\leqq P(M_a \cdot T_a^c \mid \theta) + P(M_a^c \mid \theta),$$

so that,

$$(14)\qquad \text{l.h.s. of (7)} \quad \leqq \int_{-a}^{a} P(M_a \cdot T_a^c \mid \theta)\, d\theta + \int_{-a}^{a} P(M_a^c \mid \theta)\, d\theta$$
$$\leqq \int_{-a}^{a} P(M_a \cdot T_a^c \mid \theta)\, d\theta + 4E(v(x)),$$

where the last inequality follows from (13). Now the sets M_a and T_a being defined (note (6)) by values of $\bar{x}$ and $v(x)$ only out of which $v(x)$ is by (d) distributed independently of $\bar{x}$ and θ, we can use the conditional probability distribution for given $v(x) = v$. Let $\phi(v)$ denote the cumulative distribution function of $v(x)$, i.e.

$$(15)\qquad \phi(v) = P[v(x) < v].$$

Then in the r.h.s. of (14) $P(M_a \cdot T_a^c \mid \theta) = \int_{v=0}^{v=a/2} d\phi(v) \cdot [P(T_a^c \mid \theta, v)]$, and hence by interchanging the order of integration in (14) and noting (12),

$$(16)\qquad \int_{-a}^{a} P(M_a \cdot T_a^c \mid \theta)\, d\theta = \int_{v=0}^{v=a/2} d\phi(v) \left[\int_{-a}^{a} P(T_a^c \mid \theta \cdot v)\, d\theta\right].$$

Now in (16),

$$\int_{-a}^{a} P(T_a^c \mid \theta, v)\, d\theta$$
$$(17)\qquad = \int_{a-v}^{a} P(T_a^c \mid \theta, v)\, d\theta + \int_{-a}^{-a+v} P(T_a^c \mid \theta, v)\, d\theta + \int_{-a+v}^{a-v} P(T_a^c \mid \theta, v)\, d\theta$$
$$\leqq \int_{a-v}^{a} d\theta + \int_{-a}^{-a+v} d\theta + \int_{-a+v}^{a-v} P(T_a^c \mid \theta, v)\, d\theta.$$

Further, since $P(T_a^c \mid \theta, v) = P(\bar{x} > a - v \mid \theta, v) + P(\bar{x} < -a + v \mid \theta, v)$ in the r.h.s. of (17), noting that $|\theta| \leqq a - v$

$$\int_{-a+v}^{a-v} P(T_a^c \mid \theta \cdot v)\, d\theta$$

$$= \int_{-a+v}^{a-v} P(\bar{x} > a - v \mid \theta \cdot v)\, d\theta + \int_{-a+v}^{a-v} P(\bar{x} < -a + v \mid \theta, v)\, d\theta$$

which is

$$< \int_{-\infty}^{a-v} P(\bar{x} > a - v \mid \theta \cdot v)\, d\theta + \int_{-a+v}^{\infty} P(\bar{x} < -a + v \mid \theta \cdot v)\, d\theta$$

$$= \int_{-\infty}^{a-v} d\theta \left[\int_{a-v-\theta}^{\infty} p(t)\, dt\right] + \int_{-a+v}^{\infty} d\theta \left[\int_{-\infty}^{-a+v-\theta} p(t)\, dt\right]$$

which on putting $t_1 = a - v - \theta$ in the first integral and $t_1 = \theta - (-a + v)$ in the second integral reduces to $\int_0^\infty dt_1 [\int_{t_1}^\infty p(t)\, dt] + \int_0^\infty dt_1 [\int_{-\infty}^{-t_1} p(t)\, dt]$. Hence by Condition (b) of the theorem

$$(18) \qquad \int_0^\infty dt_1 \left[\int_{t_1}^\infty p(t)\, dt\right] + \int_0^\infty dt_1 \left[\int_{-\infty}^{-t_1} p(t)\, dt\right] = K < \infty.$$

Thus from (17) and (18), $\int_{-a}^{a} P(T_a^c \mid \theta, v)\, d\theta \leqq K + 2v$, and substituting this in the r.h.s. of (16),

$$(19) \qquad \int_{-a}^{a} P(M_a \cdot T_a^c)\, dv \leqq \int_{v=0}^{v=a/2} (K + 2v)\, d\phi v$$

$$\leqq \int_{v=0}^{v=\infty} (K + 2v)\, d\phi(v) = K + 2E(v(x)).$$

Hence from (14),

$$(20) \qquad \text{l.h.s. of } (7) \leqq K + 6E(v(x)).$$

From (20), (7) and (10) follows that the r.h.s. of (10) is bounded above and being non-negative and non-decreasing it converges as $a \to \infty$ to a limit $\geqq 0$. We next show that this limit must be $= 0$.

Let d be an arbitrary number such that $0 < d < 1$, let v_1 and v_2 be non-negative number satisfying $p(-v_1) = p(-v_2)$ and let $v = \max(v_1, v_2)$. For given d and v, let $\beta_1 = \beta_1(v, d) > 0$ and $\beta_2 = \beta_2(v, d) > 0$ be defined by

$$(21) \qquad \int_{-v_2+\beta_2}^{v_1+\beta_2} p(t)\, dt = \int_{-v_2-\beta_1}^{v_1-\beta_1} p(t)\, dt = (1 - d) \cdot \int_{-v_2}^{v_1} p(t)\, dt.$$

We note that as $p(t)$ is strictly decreasing on both sides of the origin, v_1 and v_2 are uniquely determined given v. Next let

$$(22) \qquad \beta(v, d) = \max[\beta_1(v, d), \beta_2(v, d)].$$

We now define sets, W_d, $W_{d_1}^c$ and $W_{d_2}^c \subset \mathfrak{X}$ as follows:

$$W_d = [x : |\bar{x} - u(x)| > \beta(v(x), d)]$$

$$(23) \qquad W_{d_1}^c = [x : 0 \leqq (\bar{x} - u(x)) \leqq \beta(v(x), d)]$$

$$W_{d_2}^c = [x : -\beta(v(x), d) \leqq (\bar{x} - u(x)) < 0].$$

Then from (21) and (23) it follows that for any point $x \varepsilon W_d$

$$\int_{-v_2(x)+\bar{x}-u(x)}^{v_1(x)+\bar{x}-u(x)} p(t)\, dt < (1 - d) \cdot \int_{-v_2(x)}^{v_1(x)} p(t)\, dt,$$

so that,

$$\int_{-v_2(x)}^{v_1(x)} p(t)\,dt - \int_{-v_2(x)+\bar{x}-u(x)}^{v_1(x)+\bar{x}-u(x)} p(t)\,dt \geqq d\int_{-v_2(x)}^{v_1(x)} p(t)\,dt$$
$$\geqq [d/(1-d)]\int_{-v_2(x)+\bar{x}-u(x)}^{v_1(x)+\bar{x}-u(x)} p(t)\,dt.$$

Hence,

$$\text{r.h.s. of (10)} \geqq \int_{T_a \cdot W_d} L(x \mid \bar{x})\,dx\,\left[\int_{-v_2(x)}^{v_1(x)} p(t)\,dt - \int_{-v_2(x)+\bar{x}-u(x)}^{v_1(x)+\bar{x}-u(x)} p(t)\,dt\right]$$
$$\geqq [d/(1-d)]\int_{T_a \cdot W_d} L(x \mid \bar{x})\,dx\,\left[\int_{-v_2(x)+\bar{x}-u(x)}^{v_1(x)+\bar{x}-u(x)} p(t)\,dt\right].$$

Hence by (9):

(24) $$\text{r.h.s. of (10)} \geqq [d/(1-d)]\int_{-a}^{a} P(E_\theta \cdot W_d \cdot T_a \mid \theta)\,d\theta$$

Further in the l.h.s. of (7). $B_\theta \cdot T_a^c = B_\theta \cdot W_d \cdot T_a^c + B_\theta \cdot W_{d_1}^c \cdot T_a^c + B_\theta \cdot W_{d_2}^c \cdot T_a^c$ and thus we have from (7), (10) and (24),

(25) $$\int_{-a}^{a} p(B_\theta \cdot T_a^c \cdot W_d \mid \theta)\,d\theta + \int_{-a}^{a} P(B_\theta \cdot T_a^c \cdot W_{d_1}^c \mid \theta)\,d\theta$$
$$+ \int_{-a}^{a} P(B_\theta \cdot T_a^c \cdot W_{d_2}^c \mid \theta)\,d\theta \geqq \frac{d}{(1-d)}\int_{-a}^{a} P(E_\theta \cdot W_d \cdot T_a \mid \theta)\,d\theta.$$

Now by (24), (7), (10) and (20), the r.h.s. of (25) is bounded above and as it is non-negative and non-decreasing as a increases, it converges to a limit. But this implies that by making a sufficiently large, we can make arbitrarily small the difference, $\lim_{a\to\infty}[\int_{-a}^{a} P(E_\theta \cdot W_d \cdot T_a \mid \theta)\,d\theta] - \int_{-a}^{a} P(E_\theta \cdot W_d \cdot T_a \mid \theta)\,d\theta$, which equals $\int_{-a}^{a} P(E_\theta \cdot W_d \cdot T_a^c \mid \theta)\,d\theta + \int_{|\theta|>a} P(E_\theta \cdot W_d \cdot \mid \theta)\,d\theta$ which is further

$\geqq \int_{-a}^{a} P(E_\theta \cdot W_d \cdot T_a^c \mid \theta)\,d\theta \geqq \int_{-a}^{a} P(B_\theta \cdot W_d \cdot T_a^c \mid \theta)\,d\theta$, since by (6), $B_\theta \subset E_\theta$.

Hence:

(26) The first term in the l.h.s. of (25) $\to 0$ as $a \to \infty$.

For evaluating the 2nd and 3rd terms in the l.h.s. of (25) we now introduce sets F_θ, G_θ defined as:

(27) $$F_\theta = [x : \theta + v_1(x) < \bar{x} \leqq \theta + v_1(x) + \beta(v(x))],$$
$$G_\theta = [x : \theta - v_2(x) - \beta(v(x)) \leqq \bar{x} < \theta - v_2(x)].$$

Consider the set $B_\theta \cdot W_{d_1}^c$. Denoting by $I_{\theta,v(x)}$, the interval

$$[\theta - v_2(x), \theta + v_1(x)],$$

it follows from (6) and (23) that for every point $x \in B_\theta \cdot W_{d_1}^c$,

(28) $$\bar{x} \not\in I_{\theta,v(x)}; \qquad u(x) \in I_{\theta,v(x)}$$

and

(29) $$u(x) = \bar{x} - h(x) \quad \text{where} \quad \beta(v(x)) \geqq h(x) \geqq 0$$

so that, $\bar{x} - h(x) \leqq \theta + v_1(x) < \bar{x}$, and thus since by (29), $h(x) \leqq \beta(v(x))$; $\bar{x} - \beta(v(x)) \leqq \theta + v_1(x) < \bar{x}$, which by (27) implies that $x \,\varepsilon\, F_\theta$. Since every point $x \,\varepsilon\, B_\theta \cdot W_{d_1}^c$ also $\varepsilon\, F_\theta$, we have $B_\theta \cdot W_{d_1}^c \subset F_\theta$. It is similarly seen that $B_\theta \cdot W_{d_2}^c \subset G_\theta$. Hence

$$(30) \qquad \begin{aligned} P(B_\theta \cdot T_a^c \cdot W_{d_1}^c \mid \theta) &\leqq P(F_\theta \cdot T_a^c \mid \theta) \\ P(B_\theta \cdot T_a^c \cdot W_{d_2}^c \mid \theta) &\leqq P(G_\theta \cdot T_a^c \mid \theta). \end{aligned}$$

Now the sets $F_\theta \cdot T_a^c$ and $G_\theta \cdot T_a^c$ being defined by values only of $\bar{x}$, $v(x)$ and $\beta(v(x))$, and as by Condition (d) of the theorem, $v(x)$ and consequently $\beta(v(x))$, are distributed independently of $\bar{x}$ and θ, we can use the conditional probabilities for given $v(x)$. Hence by (15), $P(F_\theta \cdot T_a^c \mid \theta) = \int_{v=0}^{v=\infty} P(F_\theta \cdot T_a^c \mid \theta, v)\, d\phi(v)$, and hence by interchanging the order of integration,

$$(31) \qquad \int_{-a}^{a} P(F_\theta \cdot T_a^c \mid \theta)\, d\theta = \int_{v=0}^{v=\infty} d\phi(v)\left[\int_{-a}^{a} P(F_\theta \cdot T_a^c \mid \theta, v)\, d\theta\right].$$

Now from the Definition (27) of F_θ, $P(F_\theta \mid \theta, v) = \int_{v_1}^{v_1+\beta(v)} p(t)\, dt \leqq p(0) \cdot \beta(v)$, as $p(t)$ is strictly decreasing by (b) of the theorem. Thus in the r.h.s. of (31) for all θ,

$$(32) \qquad P(F_\theta \cdot T_a^c \mid \theta \cdot v) \leqq P(F_\theta \mid \theta \cdot v) \leqq p(0) \cdot \beta(v).$$

Now from (27), for given v, and for $x \,\varepsilon\, F_\theta$, $\theta + v_1 \leqq \bar{x} \leqq \theta + v_1 + \beta(v)$, and from (6), for $x \,\varepsilon\, T_a^c$, $\bar{x} > a - v$ or $\bar{x} < -a + v$. Therefore the intersection set $F_\theta \cdot T_a^c$ is empty unless either $\theta \leqq -a + v - v_1$ or $\theta \geqq a - v - v_1 - \beta(v)$. Thus for the values of θ in the range $(-a, a)$ the set $F_\theta \cdot T_a^c$ is empty so that $p(F_\theta \cdot T_a^c \mid \theta, v) = 0$ except for values of θ in the subranges $-a \leqq \theta \leqq -a + v - v_1$ and $a - v - v_1 - \beta(v) \leqq \theta < a$. Therefore in the r.h.s. of (31) we have

$$\int_{-a}^{a} P(F_\theta \cdot T_a^c \mid \theta, v)\, d\theta$$
$$= \int_{-a}^{-a+v-v_1} P(F_\theta \cdot T_a^c \mid \theta, v)\, d\theta + \int_{a-v-v_1-\beta(v)}^{a} P(F_\theta \cdot T_a^c \mid \theta, v)\, d\theta$$

which by (32) is $< p(0)[\int_{-a}^{-a+v-v_1} \beta(v)\, d\theta + \int_{a-v-v_1-\beta(v)}^{a} \beta(v)\, d\theta]$ which is equal to $p(0)[\beta(v)(v - v_1) + \beta(v)(v + v_1 + \beta(v)] = p(0) \cdot (2v \cdot \beta(v) + \beta^2(v))$, so that the r.h.s. of (31) is $< p(0) \int_{v=0}^{v=\infty} d\phi(v)(2v \cdot \beta(v) + \beta^2(v)) = p(0)[2E(v \cdot \beta(v)) + E(\beta^2(v))]$ thus giving

$$(33) \qquad \int_{-a}^{a} P(F_\theta \cdot T_a^c \mid \theta, v)\, d\theta \leqq p(0)[2E(v \cdot \beta(v)) + E(\beta^2(v))].$$

We next show that the expectations which occur in the r.h.s. of (33) must be finite. Since $p(t)$ is strictly decreasing on both sides of the origin, we must have $p(t) > 0$ for all t and further since $\int_{-\infty}^{\infty} p(t)\, dt = 1$, $p(t)$ must $\rightarrow 0$ when $t \rightarrow +\infty$ or $t \rightarrow -\infty$. Hence as v_1 and v_2 satisfy $p(v_1) = p(-v_2)$, as $v_1 \rightarrow +\infty$, v_2 also $\rightarrow +\infty$ and conversely. Now in the ratio $r_1 = \int_{-v_2-\beta_1}^{v_1-\beta_1} p(t)\, dt \div \int_{-v_2}^{v_1} p(t)\, dt$, put $\beta_1 = v_1$ and let $v_1 \rightarrow \infty$ so that v_2 also $\rightarrow \infty$. Then $r_1 \rightarrow \int_{-\infty}^{0} p(t)\, dt \div \int_{-\infty}^{+\infty} p(t)\, dt = c_1$ say, <1. Similarly, in the ratio, $r_2 = \int_{-v_2+\beta_2}^{v_1+\beta_2} p(t)\, dt \div \int_{-v_2}^{v_1} p(t)\, dt$. Putting $\beta_2 = v_2$ and letting $v_2 \rightarrow \infty$, so that v_1 also $\rightarrow \infty$, $r_2 \rightarrow \int_{0}^{\infty} p(t)\, dt \div \int_{-\infty}^{\infty} p(t)\, dt = c_2$ say, <1. Let $c = \max[c_1, c_2]$. Then choose d so that $1 - d > c$. Then for sufficiently large v, $\beta_1 \leqq v_1$ and $\beta_2 \leqq v_2$, so that $\beta = \max(\beta_1, \beta_2) \leqq v = \max(v_1, v_2)$. Thus we can find a number k such that for $v > k$, $\beta(v) \leqq v$. Now for $0 \leqq v \leqq k$, let m be the maximum value of $\beta(v)$. Then

$$\begin{aligned} E(v\cdot\beta(v)) &= \int_{v=0}^{v=k} v\cdot\beta(v)\, d\phi(v) + \int_{v=k}^{v=\infty} v\cdot\beta(v)\, d\phi(v) \\ &\leqq m \int_{v=0}^{v=k} v\, d\phi(v) + \int_{v=k}^{v=\infty} v^2\, d\phi(v) \\ &\leqq m\cdot E(v) + E(v^2). \end{aligned}$$

It is similarly seen that $E(\beta^2(v)) \leqq m^2 + E(v^2)$. Thus the expectations in the r.h.s. of (33) are finite.

Next by a calculation on similar lines, we obtain,

$$\int_{-a}^{a} P(G_\theta\cdot T_a^c \mid \theta)\, d\theta \leqq p(0)[2E(v\cdot\beta(v)) + E(\beta^2(v))]. \tag{34}$$

Combining (30), (33) and (34), we have, for all a,

$$\int_{-a}^{a} P(B_\theta\cdot T_a^c\cdot W_{d_1}^c \mid \theta)\, d\theta + \int_{-a}^{a} P(B_\theta\cdot T_a^c\cdot W_{d_2}^c \mid \theta)\, d\theta \leqq 2p(0)[2E(v\cdot\beta(v)) + E(\beta^2(v))] \tag{35}$$

Now from (21), it follows that for fixed v, $\beta(v, d) \to 0$ as $d \to 0$. From this it follows that each of the expectations in the r.h.s. of (35) can be made arbitrarily small. For consider $E(v, \beta(v)) = \int_{v=0}^{v=\infty} v\cdot\beta(v)\, d\phi(v)$. Since this integral is convergent, given any arbitrarily small number $\epsilon > 0$. we can find a number c such that $\int_{v=c}^{v=\infty} v\cdot\beta(v)\, d\phi \leqq \epsilon/2$. Then by making d sufficiently small we can secure that for all values of v in the range $0 \leqq v \leqq c$, $\beta(v) \leqq \epsilon/2\cdot E(v)$, so that

$$E(v, \beta(v)) = \int_{v=0}^{v=c} v\cdot\beta(v)\, d\phi(v) + \int_{v=c}^{v=\infty} v\cdot\beta(v)\, d\phi(v)$$

which is $\leqq [\epsilon/2\cdot E(v)] \int_{v=0}^{v=c} v\cdot d\phi(v) + \epsilon/2 \leqq \epsilon/2 + \epsilon/2 = \epsilon$. Similarly $E(\beta^2(v))$ can be made arbitrarily small. Thus by first making d sufficiently small, the 2nd and 3rd terms in the l.h.s. of (25) can each be made $\leqq \epsilon/3$ and then keeping d fixed by making a sufficiently large, the first term in the l.h.s. of (25) can be made $\leqq \epsilon/3$, so that the l.h.s. of (25) $\leqq \epsilon$. But the l.h.s. of (25) is the same as the l.h.s. of (7) which is $\geqq$ r.h.s. of (10). Now the r.h.s. of (10) is non-negative and non-decreasing as a increases and since it is $<$ every arbitrarily small number ϵ, it must be $= 0$ for all a. But this implies that the integrand in the r.h.s. of (10), which is everywhere non-negative, must vanish, except at most on a null subset N_a of the set of integration T_a. Since by (c) of the theorem, the factor $L(x \mid \bar{x}) > 0$ for all $x \varepsilon \mathfrak{X}$, it follows that the other factor in the integrand in the r.h.s. of (10) namely:

$$\int_{-v_2(x)}^{v_1(x)} p(t)\, dt - \int_{-v_2(x)+\bar{x}-u(x)}^{v_1(x)+\bar{x}-u(x)} p(t)\, dt = 0 \text{ for all } x \varepsilon T_a, \text{ except at most for } x \varepsilon N_a. \tag{36}$$

But since $p(v_1(x)) = p(-v_2(x))$ and $p(t)$ is strictly decreasing on both sides of the origin (36) implies that $\bar{x} - u(x) = 0$ for all $x \varepsilon (T_a - N_a)$, i.e. the set $[x \varepsilon T_a : \bar{x} \neq u(x)] =$ the null set N_a. Hence the set, $N = \bigcup_{a=1}^{\infty} N_a$, is also a null set. But N is the same as the set $[x \varepsilon \mathfrak{X}: \bar{x} \neq u(x)]$. Thus $\bar{x} = u(x)$ a.e. in $\mathfrak{X}$. Therefore, the frequency being everywhere absolutely continuous by (c) of the theorem, the strict inequality in (5) and hence in (2), cannot hold for any θ and thus the confidence intervals specified in the theorem are admissible. This completes the proof.

Corollary 3.1. *If* $u_1(x)$ *and* $u_2(x)$ *are any functions satisfying* (1) *and* (2). *then* a.e. *in* $\mathfrak{X}$, $u_1(x) = \bar{x} - v_1(x)$ *and* $u_2(x) = \bar{x} + v_2(x)$.

Proof. From (3) and (4), for all $x \varepsilon \mathfrak{X}$,

$$(37) \qquad u(x) - v_1(x) \leqq u_1(x) \quad \text{and} \quad u(x) + v_2(x) \geqq u_2(x).$$

Hence since $u(x) = \bar{x}$ a.e., and the frequency is everywhere continuous, it follows from (2) that

$$P[u_1(x) \leqq \theta \leqq u_2(x) \mid \theta] = P[u(x) - v_1(x) \leqq \theta \leqq u(x) + v_2(x) \mid \theta].$$

Hence the sets $[x: u(x) - v_1(x) < u_1(x)]$ and $[x: u(x) + v_2(x) \geqq u_2(x)]$ have zero probability for all θ, and are therefore null sets, since the frequency is everywhere >0. From this and (37), the corollary follows.

Remark. 3.1. The proof of the theorem does not require $v(x)$ to have a continuous frequency function. Hence putting $v_1(x) =$ a constant h_1, we obtain the admissibility of the set of confidence intervals of constant length, $[\bar{x} - h_1 \leqq \theta \leqq \bar{x} + h_2]$. However the admissibility of this set can be proved independently, without the restriction of the convergence of $\int_0^\infty dt_1[\int_{t_1}^\infty + \int_{-\infty}^{-t_1} p(t)\, dt]$, by a direct proof as shown in an unpublished paper of the author; alternatively as shown by the referee it can also be obtained as a simple deduction from the result proved by Farrell (1964) as follows: We define the loss functoin as

$$\begin{aligned} L(\hat{\theta}, \theta) = W(\hat{\theta} - \theta) &= 1 \text{ if } \hat{\theta} - \theta > h_1 \\ &= 1 \text{ if } \hat{\theta} - \theta < -h_2 \\ &= 0 \text{ if } -h_2 \leqq \hat{\theta} - \theta \leqq h_1 . \end{aligned}$$

The function W then satisfies Farrell's conditions, his uniqueness condition (5.2) following from the condition in (b) of Theorem 3.1 that $p(t)$ is strictly decreasing on both sides of $t = 0$. Now Farrell's result states that if $E[W(\phi(x) - \theta) \mid \theta] \leqq E[W(\bar{x} - \theta) \mid \theta]$ for all θ then, $\phi(x) = \bar{x}$ a.e. in $\mathfrak{X}$. Now putting in (1), $v_1(x) = h_1$, $v_2(x) = h_2$, take $\phi(x) = u_1(x) + h_1$. Then since by (1), $u_1(x) + h_1 + h_2 \geqq u_2(x)$,

$$\begin{aligned} E[W(\phi(x) - \theta)] &= 1 - P[\phi(x) - h_1 \leqq \theta \leqq \phi(x) + h_2 \mid \theta] \\ &= 1 - P[u_1(x) \leqq \theta \leqq u_1(x) + h_1 + h_2 \mid \theta] \\ &\leqq 1 - P[u_1(x) \leqq \theta \leqq u_2(x) \mid \theta] \end{aligned}$$

which by (2)

$$\begin{aligned} &\leqq 1 - P[\bar{x} - h_1 \leqq \theta \leqq \bar{x} + h_2 \mid \theta] \\ &= E[W(\bar{x} - \theta) \mid \theta]. \end{aligned}$$

Hence by Farrell's result, $u_1(x) + h_1 = \bar{x}$ a.e. in $\mathfrak{X}$ from which the required result immediately follows.

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