MULTIVARIATE NONPARAMETRIC SEVERAL-SAMPLE TESTS

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1. Introduction and summary. Let $(X_{j1}^{(1)}, \cdots, X_{j1}^{(p)}), \cdots, (X_{jn_j}^{(1)}, \cdots, X_{jn_j}^{(p)})$ be random samples of size n_j from the c continuous p-variate distribution functions $F_j(\mathbf{x}) = F(\mathbf{x} - \boldsymbol{\theta}_j)$ where $\mathbf{x} = (x_1, \cdots, x_p)$, $\boldsymbol{\theta}_j = (\theta_{ij}, \cdots, \theta_{pj})$ and $j = 1, \cdots, c$. This paper is concerned with the problem of testing the hypothesis $H: \boldsymbol{\theta}_1 = \boldsymbol{\theta}_2 = \cdots = \boldsymbol{\theta}_c$ against the alternative that all $\boldsymbol{\theta}_j$ are not equal on the basis of the above c samples. When we especially consider the asymptotic efficiency of test, the following alternative K is adopted,

$$K: \boldsymbol{\theta}_j = \boldsymbol{\theta} + \boldsymbol{\nu}_j/N^{\frac{1}{2}}, \qquad \boldsymbol{\nu}_j = (\nu_{ij}, \cdots, \nu_{pj}), \qquad j = 1, \cdots, c.$$

We shall develop in this paper some test procedures for the hypothesis H which are originated from the paper of Chernoff-Savage [2]. When $c = 2$, that is multivariate two-sample tests, Sugiura [7] has proposed a test statistic of Wilcoxon type. On the other hand, Puri [5] has derived univariate several-sample tests in general type including the test of Kruskal-Wallis [4]. It will be shown that the latter are corresponding to the case $p = 1$ in this paper and the former is a particular one among the case $c = 2$ of our tests.

2. Notations and definitions. NOTATIONS.

$F_j^{(k)}(x)$ = marginal distribution of the kth component of the jth distribution $F_j(\mathbf{x})$

$F_j^{(k,l)}(x, y)$ = joint marginal distribution of the kth and the lth components of the jth distribution $F_j(\mathbf{x})$

$$S_{j,n}^{(k)}(x) = (\text{number of } X_{j\alpha}^{(k)} \leqq x,\ \alpha = 1, \cdots, n)/n$$

$$S_{j,n}^{(k,l)}(x, y) = (\text{number of } (X_{j\alpha}^{(k)}, X_{j\alpha}^{(l)}) \leqq (x, y))/n$$

$$H^{(k)}(x) = \textstyle\sum_{j=1}^{c} \lambda_j F_j^{(k)}(x), \qquad S_N^{(k)}(x) = \sum_{j=1}^{c} \lambda_j S_{j,n_j(x)}^{(k)}$$

$$S_N^{(k,l)}(x, y) = \textstyle\sum_{j=1}^{c} \lambda_j S_{j,n_j x,y)}^{(k,l)}$$

where $N = \sum_{j=1}^{c} n_j$, $n_j/N = \lambda_j$, $0 < \lambda_0 \leqq \lambda_1, \cdots, \lambda_c \leqq 1 - \lambda_0 < 1$, $\lambda_0 \leqq 1/c$ and we omit the subscript j of $F_j^{(\ )}$ if it is not dependent on the jth distribution.

DEFINITION 1.

$$n_j T_j^{(k)} = \textstyle\sum_{i=1}^{N} E_i Z_{ij}, \qquad (j = 1, \cdots, c \text{ and } k = 1, \cdots, p) \tag{2.1}$$

where $Z_{ij}^{(k)} = 1$ if the ith smallest among N observations of the kth component is from the jth sample and $Z_{ij}^{(k)} = 0$ otherwise and E_i is some given constant dependent on N. Then $T_j^{(k)}$ may be expressed by the following well-known form

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$$(2.2) \qquad T_j^{(k)} = \int_{-\infty}^{\infty} J_N[S_N^{(k)}(x)]\, dS_{j,n_j}^{(k)}(x), \qquad J_N(i/N) = E_i\,.$$

Under the following assumptions, Puri has proved in [5] the asymptotic normality of $T_j^{(k)}$.

ASSUMPTION 1.

(1) $J(H) = \lim_{N\to\infty} J_N(H)$ exists for $0 < H < 1$ and is not constant

(2) $\int_{I_N} [J_N(S_N^{(k)}) - J(S_N^{(k)})]\, dS_{j,n_j}^{(k)} = o_p(1/N^{\frac{1}{2}})$, $\qquad I_N = \{x : 0 < S_N^{(k)} < 1\}$

(3) $J_N(1) = o(N^{\frac{1}{2}})$

(4) $|d^iJ(H)/dH^i| \leqq K[H(1-H)]^{-i-\frac{1}{2}+\delta}$ for $i = 0, 1, 2$ and some $\delta > 0$ and almost all x, where K is used as a generic constant.

DEFINITION 2. We define the test statistic for the hypothesis H,

$$(2.3) \qquad W = \sum_{j=1}^{c} \sum_{k,l=1}^{p} n_j \hat{A}^{kl}(T_j^{(k)} - \mu)(T_j^{(l)} - \mu)$$

where $\mu = \int_0^1 J(t)\, dt$ and $[\hat{A}^{kl}] = [\hat{A}_{kl}]^{-1}$

$$(2.4) \qquad \hat{A}_{kk} = \int_0^1 J^2(t)\, dt - \mu^2$$

$$\hat{A}_{kl} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J[S_N^{(k)}(x)] J[S_N^{(l)}(y)]\, dS_N^{(k,l)}(x, y) - \mu^2, \quad \text{for } k \neq l.$$

3. Joint asymptotic normality.

LEMMA 1. *Under the Assumption* 1, *the random vector*

$$(3.1) \qquad \mathbf{w}' = N^{\frac{1}{2}}(T_1^{(l)} - \mu_1^{(l)}, \cdots, T_c^{(l)} - \mu_c^{(l)}, \cdots, T_c^{(p)} - \mu_c^{(p)})$$

has an asymptotic normal distribution where

$$(3.2) \qquad \mu_j^{(k)} = ET_j^{(k)} = \int_{-\infty}^{\infty} J[H^{(k)}(x)]\, dF_j^{(k)}(X).$$

PROOF. Following Puri's works, it holds under Assumption 1 that $N^{\frac{1}{2}}(T_j^{(k)} - \mu_j^{(k)}) - N^{\frac{1}{2}}(B_{1j}^{(k)} + B_{2j}^{(k)})$ converges to zero in probability and $N^{\frac{1}{2}}(B_{1j}^{(k)} + B_{2j}^{(k)})$ has an asymptotic normal distribution where

$$\begin{aligned} B_{1j}^{(k)} + B_{2j}^{(k)} = & -\sum_{i\neq j}^{c} (\lambda_i/n_i) \sum_{\alpha=1}^{n_i} \{B_j^{(k)}(X_{i\alpha}) - EB_j^{(k)}(X_{i\alpha})\} \\ (3.3) \qquad & + (1/n_j) \sum_{\alpha=1}^{n_j} [J[H^{(k)}(X_{j\alpha})] - \lambda_j B_j^{(k)}(X_{j\alpha}) \\ & - EJ[H^{(k)}(X_{j\alpha})] + E\lambda_j B_j^{(k)}(X_{j\alpha})] \end{aligned}$$

$$B_j^{(k)}(x) = \int_{x_0}^{x} J[H^{(k)}(x)]\, dF_j^{(k)}(x).$$

Since we may now express (3.3) as follows

$$B_{1j}^{(k)} + B_{2j}^{(k)} = \sum_{j=1}^{c} (1/n_i) \sum_{\alpha=1}^{n_i} C_{ij}^{(k)}(X_{i\alpha}),$$

then we apply the Central Limit Theorem for the n_i independent vectors

$$\mathbf{v}_{i\alpha}' = (C_{i1}^{(1)}(X_{i\alpha}), \cdots, C_{ic}^{(1)}(X_{i\alpha}), \cdots, C_{ic}^{(p)}(X_{i\alpha})) \qquad \alpha = 1, \cdots, n_i.$$

Thus the random vector $\sum_{\alpha=1}^{n_i} \mathbf{v}_{i\alpha}'/n_i^{\frac{1}{2}}$ has a limiting normal distribution and hence

$$(3.4) \qquad \sum_{i=1}^{c} \sum_{\alpha=1}^{n_i} \mathbf{v}_{i\alpha}'/(\lambda_i n_i)^{\frac{1}{2}} = N^{\frac{1}{2}}(B_{11}^{(1)} + B_{21}^{(1)}, \cdots, B_{1c}^{(p)} + B_{2c}^{(p)})$$

has also a limiting normal distribution. The fact that the random vector $\mathbf{w}'$ is asymptotically equivalent to the right hand of (3.4) leads to Lemma 1.

LEMMA 2. *Under H or K, the asymptotic covariance matrix* $\mathbf{\Sigma}$ *of* $\mathbf{w}'$ *is given by the following* (3.5) *of rank* $p(c-1)$ *if* $|\mathbf{A}| \neq 0$.

$$\mathbf{\Sigma} = \mathbf{A} \otimes \mathbf{D} \quad \textit{where} \quad \mathbf{A} = [A_{kl}] \qquad k, l = 1, \cdots, p \tag{3.5}$$

$$A_{kk} = \int_0^1 J^2(t)\,dt - \left(\int_0^1 J(t)\,dt\right)^2 \tag{3.6}$$
$$A_{kl} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J[F^{(k)}(x)]J[F^{(l)}(y)]\,dF^{(k,l)}(x, y) - \mu^2 \quad \text{for } k \neq l$$

and

$$\mathbf{D} = [-1 + (\delta_{ij}/\lambda_i)], \qquad i, j = 1, \cdots, c; \quad \delta_{ij} = 1, \quad i = j; \qquad = 0, \quad i \neq j. \tag{3.7}$$

PROOF. First denote $\sigma_{ij}^{(kl)} = N \operatorname{Cov}[T_i^{(k)} - \mu_i^{(k)}, T_j^{(l)} - \mu_j^{(l)}]$ and express the covariance matrix $\mathbf{\Sigma}$ by the form

$$\mathbf{\Sigma} = \begin{bmatrix} (\sigma_{ij}^{(11)}) & \cdots & (\sigma_{ij}^{(1p)}) \\ \vdots & \vdots & \vdots \\ (\sigma_{ij}^{(p1)}) & \cdots & (\sigma_{ij}^{(pp)}) \end{bmatrix} \qquad i, j = 1, 2, \cdots, c.$$

Then from (3.3) for $j \neq s$,

$$\begin{aligned} N^{-1}\sigma_{js}^{(kk)} = {} & \textstyle\sum_{i\neq j,s}^{c} (\lambda_i^2/n_i)E[\bar{B}_j^{(k)}(X_{i\alpha})\bar{B}_s^{(k)}(X_{i\alpha})] \\ & - (\lambda_s/n_s)E[\bar{B}_j^{(k)}(X_{s\alpha})\{\bar{J}[H^{(k)}(X_{s\alpha})] - \lambda_s\bar{B}_s^{(k)}(X_{s\alpha})\} \\ & - (\lambda_j/n_j)E[B_s^{(k)}(X_{j\alpha})\{\bar{J}[H^{(k)}(X_{j\alpha})] - \lambda_j\bar{B}_j^{(k)}(X_{j\alpha})\}] \end{aligned} \tag{3.8}$$

where $\bar{B}(X) = B(X) - EB(X)$, $\bar{J}(X) = J(X) - EJ(X)$.
We obtained the form (3.8) after some computations noticing that since the first subscript i of X_i refers to sample from the ith distribution $F_i^{(k)}(x)$ and the second α indexes observation within the subsample, the random variables $X_{i\alpha}$'s with different i or α are independent and consequently we get for $i \neq r$ or $\alpha \neq \beta$ the following

$$E\bar{B}_j^{(k)}(X_{i\alpha})\bar{B}_s^{(k)}(X_{r\beta}) = 0, \qquad E\bar{B}_j^{(k)}(X_{i\alpha})\bar{J}[H^{(k)}(X_{r\beta})] = 0 \tag{3.9}$$
$$E\bar{J}[H^{(k)}(X_{i\alpha})]\bar{J}[H^{(k)}(X_{r\beta})] = 0.$$

Now from the following expressions

$$\bar{B}_j^{(k)}(X_{i\alpha}) = -\int_{-\infty}^{\infty} \{S_{i,1}^{(k)}(x) - F_i^{(k)}(x)\}J'[H^{(k)}(x)]\,dF_j^{(k)}(x)$$

$$\begin{aligned} & J[H^{(k)}(X_{j\alpha})] - \lambda_j\bar{B}_j^{(k)}(X_{j\alpha}) \\ & \quad = -\int_{-\infty}^{\infty} \{S_{j,1}^{(k)}(x) - F_j^{(k)}(x)\}J'[H^{(k)}(x)]\textstyle\sum_{i\neq j}^{c} \lambda_i\,dF_i^{(k)}(x). \end{aligned} \tag{3.10}$$

we may obtain that

$$
\begin{aligned}
E\bar{B}_j^{(k)}(X_{i\alpha})\bar{B}_s^{(k)}(X_{i\alpha}) &= E\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{S_{i1}^{(k)}(x) - F_i^{(k)}(x)\} \\
&\quad \cdot\{S_{i1}^{(k)}(y) - F_i^{(k)}(y)\}J'[H^{(k)}(x)]J'[H^{(k)}(y)] \\
&\quad \cdot dF_j^{(k)}(x)\, dF_s^{(k)}(y) \qquad (3.11)\\
&= 2\int_{x<y} a_i^{(k)}(x, y)\, dF_j^{(k)}(x)\, dF_s^{(k)}(y), \\
a_i^{(k)}(x, y) &= F_i^{(k)}(x)(1 - F_i^{(k)}(y)]J'[H^{(k)}(x)]J'[H^{(k)}(y)]
\end{aligned}
$$

where we note in these computations that Fubini's theorem permits the interchange of integration and expectation and for $x < y$

$$
(3.12)\quad E\{S_{i,1}^{(k)}(x) - F_i^{(k)}(x)\}\{S_{i,1}^{(k)}(y) - F_i^{(k)}(y)\} = F_i^{(k)}(x)[1 - F_i^{(k)}(y)].
$$

The second term in (3.8) is first transformed to the form

$$
\begin{aligned}
-(\lambda_s/n_s)E\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{S_{s1}^{(k)}(x) - F_s^{(k)}(x)\}\{S_{s1}^{(k)}(y) - F_s^{(k)}(y)\}J'[H^{(k)}(x)] \\
\cdot J'[H^{(k)}(y)]\, dF_j^{(k)}(x)\sum_{i\neq s}^{c}\lambda_i\, dF_i^{(k)}(y)
\end{aligned}
$$

and the value is obtained by the similar techniques to (3.11)

$$
(3.13)\qquad -(\lambda_s/n_s)2\int_{x<y} a_s^{(k)}(x, y)\, dF_j^{(k)}(x)\sum_{i\neq s}\lambda_i\, dF_i^{(k)}(y).
$$

Analogously we get

$$
(3.14)\quad \text{The third term} = -(\lambda_j/n_j)^2\int_{x<y} a_j^{(k)}(x, y)\, dF_s^{(k)}(x)\sum_{i\neq j}\lambda_i\, dF_i^{(k)}(y).
$$

Substituting (3.11) $\sim$ (3.14) in (3.8),

$$
\begin{aligned}
\sigma_{j,s}^{(k,k)} &= 2\sum_{i\neq j,s}^{c}\lambda_i\int_{x<y} a_i^{(k)}(x, y)\, dF_j^{(k)}(x)\, dF_s^{(k)}(y) \\
&\quad - 2\int_{x<y} a_s^{(k)}(x, y)\, dF_j^{(k)}(x)\sum_{i\neq s}\lambda_i\, dF_i^{(k)}(y) \qquad (3.15)\\
&\quad - 2\int_{x<y} a_j^{(k)}(x, y)\, dF_s^{(k)}(x)\sum_{i\neq j}\lambda_i\, dF_i^{(k)}(y).
\end{aligned}
$$

By a similar argument noticing the identity $ES_{i1}^{(k)}(x)S_{i1}^{(l)}(y) = F_i^{(kl)}(x, y)$, we also get

$$
\begin{aligned}
(3.16)\quad \sigma_{jj}^{(kl)} &= \sum_{i\neq j}\lambda_i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_i^{(kl)}(x, y)\, dF_j^{(k)}(x)\, dF_j^{(l)}(y) \\
&\quad + (1/\lambda_j)\sum_{i\neq j, r\neq j}\lambda_i\lambda_r\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_j^{(kl)}(x, y)\, dF_i^{(k)}(x)\, dF_r^{(l)}(y)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{js}^{(kl)} &= \sum_{i\neq j,s}\lambda_i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_i^{(kl)}(x, y)\, dF_j^{(k)}(x)\, dF_s^{(l)}(y) \\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_s^{(kl)}(x, y)\, dF_j^{(k)}(x)\sum_{i\neq s}\lambda_i\, dF_i^{(l)}(y) \qquad (3.17)\\
&\quad - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a_j^{(kl)}(x, y)\, dF_s^{(l)}(y)\sum_{i\neq j}\lambda_i\, dF_i^{(k)}(x)
\end{aligned}
$$

where $a_i^{(kl)}(x, y) = \{F_i^{(kl)}(x, y) - F_i^{(k)}(x)F_i^{(l)}(y)\}J'[H^{(k)}(x)]J'[H^{(l)}(y)]$. The variance $\sigma_{jj}^{(kk)}$ of $N^{\frac{1}{2}}T_j^{(k)}$ has been derived by Puri

$$
\begin{aligned}
\sigma_{jj}^{(kk)} &= 2\sum_{i\neq j}\lambda_i\int_{x<y} a_i^{(k)}(x, y)\, dF_j^{(k)}(x)\, dF_j^{(k)}(y) + (2/\lambda_j) \\
&\quad \cdot\sum_{i\neq j}\lambda_i^2\int_{x<y} a_i^{(k)}(x, y)\, dF_i^{(k)}(x)\, dF_i^{(k)}(y) \qquad (3.18)
\end{aligned}
$$

$$+ (1/\lambda_j) \textstyle\sum_{i,r\neq j, i\neq r} \lambda_i\lambda_r[\int_{x<y} a_j^{(k)}(x, y)\, dF_i^{(k)}(x)\, dF_r^{(k)}(y)$$
$$+ \textstyle\int_{x<y} a_j^{(k)}(x, y)\, dF_i^{(k)}(y)\, dF_r^{(k)}(x)].$$

Now under the hypothesis H or K, we get the following after some computations

$$2 \lim_{N\to\infty} \textstyle\int_{x<y} a_j^{(k)}(x, y)\, dF_i^{(k)}(x)\, dF_r^{(k)}(y)$$
$$= 2 \textstyle\int_{x<y} F^{(k)}(x)[1 - F^{(k)}(y)]J'[F^{(k)}(x)]$$
$$\cdot J'[F^{(k)}(y)]\, dF^{(k)}(y) = A_{kk},$$
$$\lim_{N\to\infty} \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} a_j^{(kl)}(x, y)\, dF_i^{(k)}(x)\, dF_s^{(l)}(y) = A_{kl}$$

where A_{kl}'s are defined in (3.6). Thus we get under H or K

$$\sigma_{ij}^{(kl)} = (-1 + (\delta_{ij}/\lambda_i))A_{kl} \quad k, l = 1, \cdots, p \text{ and } i, j = 1, \cdots, c. \tag{3.19}$$

Secondly some elementary computations show that

$$|\mathbf{D}| = 0, \qquad |\mathbf{D}_1| = \lambda_c/\lambda_1 \cdots \lambda_{c-1} \neq 0$$

where $\mathbf{D}_1$ is a submatrix obtained from $\mathbf{D}$ by deleting the cth row and column. From the identities above, all the cofactors of order $pc - r$ $(r = 0, 1, \cdots, p - 1)$ of $\mathbf{\Sigma}$ vanish and a cofactor of the order $p(c - 1)$

$$\begin{vmatrix} A_{11}\mathbf{D}_1 & \cdots & A_{1p}\mathbf{D}_1 \\ \vdots & & \vdots \\ A_{p1}\mathbf{D}_1 & \cdots & A_{pp}\mathbf{D}_1 \end{vmatrix} = |\mathbf{A}|\ (\lambda_c/\lambda_1 \cdots \lambda_{c-1})^p$$

is not zero if $|\mathbf{A}| \neq 0$. Thus if follows that rank $(\mathbf{\Sigma}) = p(c - 1)$.

LEMMA 3. *Suppose that* $|J(u)| \leqq K[u(1 - u)]^{-\alpha}$, $0 < \alpha < \frac{1}{8}$, $|J'(u)| \leqq K[u(1 - u)]^{-1}$ $|J''(u)| \leqq K[u(1 - u)]^{-2}$, *then for* $k \neq l$

$$\hat{A}_{kl} = \textstyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J[S_N^{(k)}(x)]J[S_N^{(l)}(y)]\, dS_N^{(kl)}(x, y) - \mu^2 \tag{3.20}$$

is a consistent estimator for A_{kl} and hence $\hat{A}^{kl}$ *is also for* A^{kl}.

The proof follows from Theorem 1 in Bhuchongkul [1]

4. The asymptotic distribution of the test statistic.

THEOREM 1. *Under the hypothesis* H, *the asymptotic distribution of* $\mathbf{w}'\mathbf{\Lambda}\mathbf{w}$ *is central* $\chi^2_{p(c-1)}$ *with degree of freedom* $p(c - 1)$ *where*

$$\mathbf{\Lambda} = \mathbf{A}^{-1} \otimes \mathbf{\Gamma}, \qquad \mathbf{\Gamma} = [\lambda_i\delta_{ij}] \qquad i, j = 1, 2, \cdots, c. \tag{4.1}$$

In order to prove this Theorem 1, we apply the following lemma which has been established by Sugiura [6],

LEMMA 4 (Sugiura). *Suppose that the distribution of the c-variate column vector* $\mathbf{x}$ *is normal with mean vector* $\mathbf{0}$ *and the covariance matrix* $\mathbf{\Sigma}$ *of rank* $r(\leqq c)$. *Then there exists an unique* $c \times c$ *matrix* $\mathbf{\Lambda}$ *such that*

$$\mathbf{B\Lambda} = \mathbf{0}, \qquad \mathbf{\Sigma\Lambda} = \mathbf{I} - \mathbf{B} \tag{4.2}$$

where $\mathbf{B}$ *is the projection of the c-dimensional Euclidean vector space to the eigen-*

space belonging to the eigenvalue zero of $\mathbf{\Sigma}$. *This* $\mathbf{A}$ *is symmetric and* $\mathbf{x}'\mathbf{A}\mathbf{x}$ *is distributed as* χ_r^2.

PROOF OF THEOREM 1. $\mathbf{\Sigma}$ is given in (3.5) and we easily get

$$\text{(4.3)} \qquad \mathbf{B} = (1/\alpha)\begin{bmatrix} \mathbf{B}_1 & \cdots & \mathbf{B}_1 \\ \vdots & & \vdots \\ \mathbf{B}_1 & \cdots & \mathbf{B}_1 \end{bmatrix}, \qquad \mathbf{B}_1 = [\lambda_i\lambda_j], \qquad \alpha = \sum_{j=1}^{c}\lambda_j^2$$

$$i, j = 1, \cdots, c.$$

Let $\mathbf{A} = (x_{ij})\ i, j = 1, \cdots, c, \cdots, pc$, then the first equation of (4.2) is expressed for all j, $\sum_{k=1}^{p}\sum_{i=1}^{c}\lambda_i x_{(k-1)c+1,j} = 0$. Then we restrict to x_{ij} satisfying the following equations

$$\text{(4.4)} \qquad \sum_{i=1}^{c}\lambda_i x_{(k-1)c+1,j} = 0 \qquad \text{for all } j \text{ and } k.$$

From the second part of (4.2), we get the following equations for the variables x_{ij} from the first column to the cth of $\mathbf{A}$

$$\text{(4.5)} \qquad \begin{aligned} -A_{11}\sum x_{ij} + (A_{11}/\lambda_i)x_{ij} - A_{12}\sum x_{c+1,j} + (A_{12}/\lambda_i)x_{c+i,j} - \cdots \\ - A_{1p}\sum x_{(p-1)c+1,j} + (A_{1p}/\lambda_i)x_{(p-1)c+i,j} = \delta_{ij} - (\lambda_i\lambda_j/\alpha) \\ -A_{k1}\sum x_{ij} + (A_{k1}/\lambda_i)x_{ij} - \cdots - A_{kp}\sum x_{(p-1)c+i,j} \\ + (A_{kp}/\lambda_i)x_{(p-1)c+i,j} = -\lambda_i\lambda_j/\alpha, \end{aligned}$$

for $k = 2, \cdots, p$; $i, j = 1, 2, \cdots, c$ and $\sum$ means $\sum_{i=1}^{c}$. Multiplying λ_i^2 on both sides of (4.5) and summing up with regard to i noticing (4.4),

$$-A_{k1}\sum x_{ij} - \cdots - A_{kp}\sum x_{(p-1)c+i,j} = (\delta_{k1}/\alpha)\lambda_j^2 - (\beta/\alpha^2)\lambda_j \quad k = 1, 2, \cdots, p.$$

Substituting these values in (4.5), we get

$$A_{11}x_{ij} + \cdots + A_{1p}x_{(p-1)c+i,j} = \lambda_i\delta_{ij} - [\lambda_i\lambda_j(\lambda_i + \lambda_j)/\alpha] + (\beta/\alpha^2)\lambda_i\lambda_j$$

$$A_{k1}x_{ij} + \cdots + A_{kp}x_{(p-1)c+i,j} = -(\lambda_i^2\lambda_j/\alpha) + (\beta/\alpha^2)\lambda_i\lambda_j, \quad k = 2, \cdots, p.$$

Under $|\mathbf{A}| \neq 0$, we may solve the above equation as $x_{(k-1)c+i,j} = \lambda_i\delta_{ij}A^{ik} + a\lambda_i\lambda_j(\lambda_i + \lambda_j + b)$, where $[A^{kl}] = [A_{kl}]^{-1}$, $k, l = 1, 2, \cdots, p$ and a, b are generic constants. Analogously we get

$$\text{(4.6)} \qquad x_{(k-1)c+i,(l-1)c+j} = \lambda_i\delta_{ij}A^{lk} + a\lambda_i\lambda_j(\lambda_i + \lambda_j + b).$$

By remarking the following facts

(1) $\sum_{j=1}^{c}\lambda_j(T_j^{(k)} - \mu_j^{(k)}) = 0$ for all k

(2) we concern with only the form of $\mathbf{w}'\mathbf{A}\mathbf{w}$,

we may neglect the term $a\lambda_i\lambda_j(\lambda_i + \lambda_j + b)$ in the form (4.6) of x_{ij} and hence get the form (4.1).

Now we propose the following test statistic

$$\text{(4.7)} \qquad W = \sum_{j=1}^{c}\sum_{k,l=1}^{p} n_j\hat{A}^{kl}(T_j^{(k)} - \mu)(T_j^{(l)} - \mu).$$

THEOREM 2. *Under the hypothesis H, W has a limiting* $\chi^2_{p(c-1)}$ *distribution.*

The proof easily follows from Lemma 3 and Theorem 1.

THEOREM 3. *Suppose that the density* $f^{(k)}(x)$ *of* $F^{(k)}(x)$ *be bounded and* $dJ[F^{(k)}(x)]/dx$ *be also bounded as* $x \to \pm\infty$. *Then under K, the asymptotic distribution of W is noncentral* $\chi^2_{p(c-1)}(\tau)$ *with noncentrality parameter* τ *given by*

$$(4.8) \quad \tau = \sum_{j=1}^{c} \sum_{k,l=1}^{p} \lambda_i(\nu_{kj} - \bar{\nu}_k)(\nu_{lj} - \bar{\nu}_l)A^{kl} \cdot [\int_{-\infty}^{\infty} J'[F^{(k)}(x)]f^{(k)}(x)\, dF^{(k)}(x)][\int_{-\infty}^{\infty} J'[F^{(l)}(x)]f^{(l)}(x)\, dF^{(l)}(x)]$$

where $\bar{\nu}_k = \sum_{i=1}^{c} \lambda_i \nu_{ki}$.

Proof. From Lemma 2 and Theorem 1, asymptotic noncentral $\chi^2_{p(c-1)}$ distribution of W is easily shown under the alternative K where non-central parameter τ is given by $\tau = \lim_{N\to\infty} \sum_{j=1}^{c} \sum_{k,l=1}^{p} NA^{kl}(\mu_j^{(k)} - \mu)(\mu_j^{(l)} - \mu)$. On the other hand, Hodges-Lehmann [3] and supposition of Theorem 3 lead that

$$\mu_j^{(k)} - \mu = \int_{-\infty}^{\infty} \{J(\sum_{i=1}^{c} \lambda_i F^{(k)}(x + (\nu_{kj} - \nu_{ki}/N^{\frac{1}{2}}))) - J[F^{(k)}(x)]\}\, dF^{(k)}(x) \sim N^{-\frac{1}{2}}(\nu_{kj} - \bar{\nu}_k) \int_{-\infty}^{\infty} J'[F^{(k)}(x)]f^{(k)}(x)\, dF^{(k)}(x).$$

Thus we obtain (4.8).

5. Some discussions.

First we consider the case $c = 2$ as a special one. Then (4.8) reduce to

$$(5.1) \quad \tau = \lambda_1\lambda_2 \sum_{k,l=1}^{p} \nu_k\nu_l A^{kl}[\int_{-\infty}^{\infty} J'[F^{(k)}(x)]^{(k)}(x)\, dF^{(k)}(x)] \cdot [\int_{-\infty}^{\infty} J'[F^{(l)}(x)]f^{(l)}(x)\, dF^{(l)}(x),$$

where $\nu_k = \nu_{k1} - \nu_{k2}$. Then the asymptotic efficiency of the test W with respect to the Hotelling T^2-test is given by

$$(5.2) \quad e_{W,T^2} = \sum_{k,l} \nu_k\nu_l A^{kl}[\int_{-\infty}^{\infty} J'[F^{(k)}(x)]f^{(k)}(x)\, dF^{(k)}(x)] \cdot [\int_{-\infty}^{\infty} J'[F^{(l)}(x)]f^{(l)}(x)\, dF^{(l)}(x)]/\sum_{k,l} \nu_k\nu_l\sigma^{kl}$$

where $\sigma_{kl} = \text{Cov}(X^{(k)}, X^{(l)})$ and $[\sigma^{kl}] = [\sigma_{kl}]^{-1}$. Putting $J(u) = u$, which is corresponding to the Wilcoxon test, we get

$$(5.3) \quad e_{W,T^2} = \sum_{k,l} \nu_k\nu_l A^{kl}[\int_{-\infty}^{\infty} f^{(k)}(x)\, dF^{(k)}(x)][\int_{-\infty}^{\infty} f^{(l)}(x)\, dF^{(l)}(x)]/\sum_{k,l} \nu_k\nu_l\sigma^{kl}$$

which is consistent in the result by Sugiura [7]. Secondly we set $J(u) = \Phi^{-1}(u)$, which constructs the normal scores test, and we assume that the underlying distribution be normal $F(\mathbf{x}) = \Phi(\mathbf{0}, [\sigma_{ij}])$, then we easily get

$$A_{kl} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, d\Phi^{(kl)}(x, y) - (\int_{-\infty}^{\infty} x\, d\Phi^{(k)}(x))^2 = \sigma_{kl}$$

and

$$\int_{-\infty}^{\infty} J'[\Phi^{(k)}(x)]\phi^{(k)}(x)\, d\Phi^{(k)}(x) = 1.$$

Therefore it follows from (5.2) that $e_{W,T^2} = 1$. Lastly consider the case $p = 1$, that is univariate c-sample test. It follows from (4.8) that

$$\tau = A_{11} \sum_{j=1}^{c} \lambda_j(\nu_j - \bar{\nu})^2[\int_{-\infty}^{\infty} J'[F(x)]f(x)\, dF(x)]^2$$

which is also equivalent to Puri's result.

After the present paper had been submitted, the author learned that the same results were obtained independently by M. L. Puri and P. K. Sen, and, for the case $c = 2$, by G. K. Bhattacharyya. An abstract of Bhattacharyya's work appears in *Ann. Math. Statist.* **36** (1965) 1905–1906.

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