DUALS OF PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS AND SOME NONEXISTENCE THEOREMS[1]

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1. Introduction and summary. In this note, we give simpler proofs of results obtained, regarding the necessary conditions for existence of certain unsymmetrical PBIB designs having group divisible, triangular and L_i association schemes, in [5]. We also give simpler proofs of the necessary conditions for the existence of affine α-resolvable $GD(m, n)$, $T(n)$, $L_i(s)$ and BIB designs obtained in [6]. We follow the notation of these two papers throughout this note. We finally obtain necessary conditions for the dual of a design D, to be a specified design E.

2. Gramians of the rational characteristic vectors of $N'N$ in terms of the gramians of rational characteristic vectors of NN'. Let $\theta_0 = rk, \theta_1, \theta_2, \cdots, \theta_s$ be the distinct rational characteristic roots of NN', where N is the incidence matrix of the design defined in the usual manner and N' is its transpose, with respective multiplicities $\alpha_0 = 1, \alpha_1, \alpha_2, \cdots, \alpha_s$ and zero be a characteristic root with multiplicity β. We shall put $\beta = 0$ if NN' is non singular and the results of this note still hold true. Obviously $1 + \sum_{i=1}^{s} \alpha_i + \beta = v$. Then, the characteristic roots of $N'N$ are $\theta_0, \theta_1, \theta_2, \cdots, \theta_s, 0$ with respective multiplicities $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_s, \beta + b - v$.

Let $\underset{\sim}{x}_{i1}, \underset{\sim}{x}_{i2}, \cdots, \underset{\sim}{x}_{i\alpha_i}$ be a set of independent, rational characteristic vectors corresponding to the root θ_i of NN'. Putting

$$X = [\underset{\sim}{x}_{i1}, \underset{\sim}{x}_{i2}, \cdots, \underset{\sim}{x}_{i\alpha_i}], \tag{2.1}$$

the gramian of the independent, rational characteristic vectors corresponding to the root θ_i of NN' is, by definition,

$$Q_i = X_i'X_i, \qquad i = 1, 2, \cdots, s. \tag{2.2}$$

Since the column vectors of X_i are the rational characteristic vectors of NN' corresponding to the root θ_i, it follows that the column vectors of $N'X_i$ are the independent, rational characteristic vectors of $N'N$ corresponding to the same root θ_i. Hence the gramian of the independent, rational characteristic vectors corresponding to the root θ_i of $N'N$ is

$$Q_i^* = X_i'NN'X_i = X_i'\theta_iX_i = \theta_iQ_i, \qquad i = 1, 2, \cdots, s. \tag{2.3}$$

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Hence

$$|Q_i^*| = \theta_i^{\alpha_i} |Q_i|, \tag{2.4}$$

and from Lemma 1.4 of Ogawa [4],

$$c_p(Q_i^*) = (-1, \theta_i)_p^{\alpha_i(\alpha_i+1)/2}(\theta_i, |Q_i|)_p^{\alpha_i-1}c_p(Q_i), \tag{2.5}$$

where $(a, b)_p$ is the Hilbert norm residue symbol and $c_p(A)$ is the Hasse-Minkowski invariant of A.

In the above discussion we have considered the gramians of the independent, rational characteristic vectors corresponding to the non zero, rational roots only. If Q and Q^* are the gramians of the independent rational, characteristic vectors corresponding to the zero root of NN' and $N'N$ respectively, then $|Q|$, $|Q^*|$, $c_p(Q)$ and $c_p(Q^*)$ can be evaluated with the help of Equations (3.11) and (3.13) of [5]. In what follows p stands for odd primes only.

3. Simpler proofs of results of [5]. A semi-regular $GD(m, n)$ satisfying $b = v - m + 1$ is a LB design (Cf. Laha and Roy [3]). Let N be the incidence matrix of such design. Then the non zero characteristic roots of NN' are $\theta_0 = rk$ and $\theta_2 = r - \lambda_1$, with respective multiplicities $\alpha_0 = 1$ and $\alpha_2 = m(n - 1)$ (cf. Connor and Clatworthy [2]). It can be easily shown that Q_2 has

$$|Q_2| \sim n^m, \tag{3.1}$$

and

$$c_p(Q_2) = (-1, n)_p^{m(m+3)/2}, \tag{3.2}$$

where $a \sim b$ stands for the fact that the square free parts of a and b are the same. Then Q_2^*, in view of (2.4) and (2.5) has

$$|Q_2^*| \sim \theta_2^{\alpha_2} n^m, \tag{3.3}$$

and

$$c_p(Q_2^*) = (-1, \theta_2)_p^{\alpha_2(\alpha_2+1)/2}(\theta_2, n^m)_p^{\alpha_2-1}(-1, n)_p^{m(m+3)/2}. \tag{3.4}$$

Since the semi-regular $GD(m, n)$ is a LB design, Q_2^* must be rational congruent to the gramian corresponding to the independent, rational characteristic vectors corresponding to the multiple root (i.e. θ_2) of the BIB design, which is the dual of the semi-regular $GD(m, n)$ under consideration. Hence

$$|Q_2^*| \sim b, \tag{3.5}$$

and

$$c_p(Q_2^*) = 1. \tag{3.6}$$

Hence, if the semi-regular $GD(m, n)$ with $b = v - m + 1$ exists, then from (3.3) and (3.5), we must have

$$\theta_2^{\alpha_2} n^m b \sim 1 \tag{3.7}$$

and from (3.4) and (3.6) we must have

$$(-1, \theta_2)_p^{\alpha_2(\alpha_2+1)/2} (\theta_2, n^m)_p^{\alpha_2-1} (-1, n)_p^{m(m+3)/2} = 1. \tag{3.8}$$

We can easily verify that Conditions (3.7) and (3.8) are equivalent to the conditions given in Theorem 4.1 of [5]. Theorems 5.1, 5.2, 6.1 and 6.2 of [5] can similarly be proved.

4. Simpler proofs of results of [6]. Let N be the incidence matrix of an affine α-resolvable BIB design with parameters v, $b = t\beta$, $r = t\alpha$, k and λ. The gramian Q_1 of the rational characteristic vectors corresponding to the root $r - \lambda$ of NN', has

$$|Q_1| \sim v, \tag{4.1}$$

and

$$c_p(Q_1) = 1. \tag{4.2}$$

Then Q_1^*, in view of (2.4) and (2.5), has

$$|Q_1^*| \sim (r - \lambda)^{v-1} v, \tag{4.3}$$

and

$$c_p(Q_1^*) = (-1, r - \lambda)_p^{v(v-1)/2} (r - \lambda, v)_p^v. \tag{4.4}$$

Since the design under consideration is affine α-resolvable, its dual is $GD(t, \beta)$ and Q_1^*, must be rationally congruent to the gramian of the independent, rational characteristic vectors corresponding to the non zero multiple root of $GD(t, \beta)$. Hence

$$|Q_1^*| \sim \beta^t, \tag{4.5}$$

and

$$c_p(Q_1^*) = (-1, \beta)_p^{t(t+3)/2}. \tag{4.6}$$

Hence if the affine α-resolvable BIB design exists, we must have

$$(r - \lambda)^{v-1} v \beta^t \sim 1, \tag{4.7}$$

and

$$(-1, r - \lambda)_p^{v(v-1)/2} (r - \lambda, v)_p^v (-1, \beta)_p^{t(t+3)/2} = 1. \tag{4.8}$$

We can easily verify that the Conditions (4.7) and (4.8) are equivalent to the necessary conditions given in Theorems 5 of [6]. Theorems 8 and its corollaries of [6] can similarly be proved.

5. Duals of designs. Given two designs D and E, one may be interested to know whether the design E can be the dual of D. For this purpose, we must necessarily have the number of treatments of D to be equal to the number of blocks in E, number of blocks of D to be equal to the number of treatments in E, and the number of replications of D to be equal to the block size in E. Further, if N is the incidence matrix of D and N^* is the incidence matrix of E, then the non zero characteristic roots along with their multiplicities should be the same for NN' and $N^*N^{*\prime}$. Additional necessary conditions for this purpose can be obtained in view of Section 2. For the given design D, we know the gramians Q_i of the independent, rational characteristic vectors corresponding to the roots θ_i of $NN'(i = 1, 2, \cdots, s)$. If Q_i^* is the gramian of the independent rational characteristic vectors corresponding to the root θ_i of $N'N$, then $|Q_i^*|$ and $c_p(Q_i^*)$ can be evaluated from (2.4) and (2.5). For the design E, the values of $|P_i|$ and $c_p(P_i)$ can be independently obtained, where P_i is the gramian of the independent, rational characteristic vectors corresponding to the characteristic root θ_i of $N^*N^{*\prime}$ $(i = 1, 2, \cdots, s)$. In order that E be the dual of D, it is clearly necessary that P_i be rationally congruent to Q_i^* and hence $|Q_i^*| \sim |P_i|$ and $c_p(Q_i^*) = c_p(P_i)$ $(i = 1, 2, \cdots, s)$.

Let us consider the particular case of symmetrical designs, i.e. designs in which $v = b$ and $r = k$. We know that the dual of a symmetrical BIB design is again a symmetrical BIB design. We may be interested to know whether a similar property holds in case of PBIB designs.

Let N be the incidence matrix of a symmetrical PBIB design with association scheme A and Q_i be the gramian of the independent, rational characteristic vectors corresponding to the characteristic root θ_i of $NN'(i = 1, 2, \cdots, s)$. If Q_i^* is the gramian of the independent, rational characteristic vectors corresponding to the root θ_i of $N'N$, then (2.4) and (2.5) give the value of $|Q_i^*|$ and $c_p(Q_i^*)$ respectively.

If the dual of this design is again a PBIB design with the same set of parameters and association scheme A, it is necessary that

$$|Q_i^*| \sim |Q_i| \tag{5.1}$$

and

$$c_p(Q_i^*) = c_p(Q_i). \tag{5.2}$$

Hence in order that the dual of a symmetrical PBIB design with association scheme A to be a PBIB design with the same set of parameters and the same association scheme A, it is necessary that

$$\theta_i^{\alpha_i} \sim 1, \qquad i = 1, 2, \cdots, s, \tag{5.3}$$

and

$$(-1, \theta_i)_p^{\alpha_i(\alpha_i+1)/2}(\theta_i, |Q_i|)_p^{\alpha_i - 1} = 1, \qquad i = 1, 2, \cdots, s. \tag{5.4}$$

Thus

THEOREM 5.1. *Necessary conditions for the dual of a symmetrical* PBIB *design* *with association scheme A to be a* PBIB *design with the same set of parameters and the same association scheme A are* (5.3) *and* (5.4).

Illustration 5.1.1. Consider $T(2)$ of the Tables [1] with parameters

$$v = 10 = b, \qquad r = 4 = k, \qquad n = 5, \qquad \lambda_1 = 2, \qquad \lambda_2 = 0. \tag{5.5}$$

From Corollary 5.1.4 of Connor and Clatworthy [2], we can obtain

$$\theta_1 = 6, \qquad \alpha_1 = 4, \qquad |Q_1| \sim 5. \tag{5.6}$$

The left hand side of (5.4) becomes

$$(5.6)_p \tag{5.7}$$

which has the value -1 for $p = 3$. Hence the Condition (5.4) is violated. Thus the dual of $T(2)$ with parameters (5.5) cannot be a $T(2)$ with parameters (5.5). In fact the dual of this design is the singular group divisible design designated by S 17.

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REFERENCES

[1] BOSE, R. C., CLATWORTHY, W. H. and SHRIKHANDE, S. S. (1954). Tables of partially balanced incomplete block designs, Inst. of Statist., Univ. of North Carolina, Reprint Series No. 50.

[2] CONNOR, W. S. and CLATWORTHY, W. H. (1954). Some theorems for partially balanced designs. *Ann. Math. Statist.* **25** 100–112.

[3] LAHA, R. G. and ROY, J. (1956–57). Classification and analysis of linked block designs. *Sankhyā* **17** 115–132.

[4] OGAWA, JUNJIRO (1959). A necessary condition for existence of regular symmetrical experimental designs of triangular type with partially balanced incomplete blocks. *Ann. Math. Statist.* **30** 1063–71.

[5] SHRIKHANDE, S. S., RAGHAVARAO, D. and THARTHARE, S. K. (1963). Non existence of some unsymmetrical partially balanced incomplete block designs, *Canad. J. Math.* **15** 686–701.

[6] SHRIKHANDE, S. S. and RAGHAVARAO, D. (1963). Affine α-resolvable incomplete block designs. Contributions to Statistics volume presented to Professor P. C. Mahalanobis on his 70th birthday. Pergamon Press, New York, 471–480.