SINGULARITY IN HOTELLING'S WEIGHING DESIGNS AND A GENERALIZED INVERSE

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1. Summary and introduction. Results of N weighing operations to determine the individual weights of p objects, as envisaged in Hotelling's weighing designs [4], fit into the linear model $Y = X\beta + e$, where Y is an $N \times 1$ random observed vector of the recorded results of weighings; $X = (x_{ij})$, $i = 1, 2, \cdots, N$, $j = 1, 2, \cdots, p$, is an $N \times p$ matrix of known quantities with $x_{ij} = +1, -1$ or 0, if, in the ith weighing operation, the jth object is placed respectively in the left pan, right pan, or in none; β is a $p \times 1$ vector ($p \leqq N$) representing the weights of the objects; e is an $N \times 1$ unobserved random vector such that $E(e) = 0$ and $E(ee') = \sigma^2 I_N$. X represents the weighing design matrix. When X is of full rank, that is, when $X'X$ is non-singular, the weights of the objects are given by the least squares estimates, $\hat{\beta} = [X'X]^{-1}X'Y$. The covariance matrix is given by $\operatorname{Cov}(\hat{\beta}) = \sigma^2[X'X]^{-1} = \sigma^2 C$. c_{ii}, which is the ith diagonal element of C, represents the variance factor for the ith object. In weighing designs, we search for the elements x_{ij} such that c_{ii} is the least for each i.

When, however, X is singular, it is well known that, while it is not possible to have a unique (unbiased) estimate for each of the p objects, it is possible to have a unique (unbiased) estimate for a linear function, $\lambda'\beta$, of the parameters, if and only if there exists a solution for r in the equations $Sr = \lambda$, where $S = X'X$. Raghavarao [7] visualized that *bad designing*, *repetitions* or *accidents* might lead to singular weighing designs, and considered the question of taking additional weighings to make the resultant design matrix X of full rank maximizing the resultant det. $|X'X|$, if possible, as required in Mood's [5] efficiency definition. He appeared to take only the chemical balance (and not the spring balance) into consideration, and was eventually led to the question of dealing with the situation when the rank of X was less than the full by only one.

In this paper, the problem is, in the first place, cast into the framework of a generalized inverse (referred to as a g-inverse). Two harmonizing results in the area of g-inverses are then indicated by way of an aid to algebraic simplifications in the context of tackling the general problem. Singular weighing designs may not perhaps all be altogether useless in themselves, although one may not adopt singular weighing designs in a scheme of weighing operations to begin with.

In certain given situations, singular weighing designs may even be preferred to the best available weighing designs adopted for the estimation of the weight of each individual object, and, in such situations, there may arise a necessity for comparing two singular weighing designs. Linking the problem of singular weighing designs to a g-inverse, as has been done in this paper, would also help

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institute such a comparison. Consideration is further given to the situation when the rank of X may be less than the full by more than one.

Finally, a special class of singular spring balance designs is also discussed.

2. Generalized inverse. In an attempt to provide a unified approach to least squares theory with reference to the case when the matrix of normal equations becomes singular, Rao [8] has suggested two computations of a g-inverse. (In this context, the reader may also be referred to Price [6] and Zelen [10].) The computation of one is exactly the same as that of a regular inverse when it exists. The method of "sweep out" is applied to the matrix A for which an inverse is required, and the same operations are applied to an appended unit matrix, until A reduces to H, and the unit matrix to B, such that $BA = H$, H is idempotent, $AH = A$, and $ABA = A$, where H has the form,

$$H = \left[\begin{array}{c|c} I & H_{12} \\ \hline 0 & 0 \end{array}\right]. \tag{2.1}$$

In (2.1), H_{12} is the reduced part of the corresponding portion of the matrix A on which the "sweep out" is applied. Here, B is a g-inverse. This inverse will be referred to, in this paper, as the first g-inverse. [Incidentally, it may be pointed out that a general solution of the equations $Ax = y$ is given by $x = By + (H - I)Z$, where Z is an arbitrary column vector.]

Rao [8] mentions about a second g-inverse which always exists, although it may not be unique. Given a matrix A, there exist non-singular matrices P_* and Q_* such that $P_*AQ_* = \Delta$, $A = P_*^{-1}\Delta Q_*^{-1}$, where

$$\Delta = \left[\begin{array}{c|c} D_r & 0 \\ \hline 0 & 0 \end{array}\right],$$

and D_r is a diagonal matrix of order r and rank r. Then, a g-inverse for A is defined as $A^- = Q_*\Delta^- P_*$, where

$$\Delta^- = \left[\begin{array}{c|c} D_r^{-1} & 0 \\ \hline 0 & 0 \end{array}\right].$$

The g-inverse is such that $AA^-A = A$ and $A^-AA^- = A^-$. This g-inverse will be referred to as the second g-inverse. When A is symmetric, Q_* will be P_*', and A^- will also be symmetric. Using this second form of the g-inverse, the following well known lemma may be easily proved.

LEMMA 2.1. *If the linear function $\lambda'\beta$ is estimable, then the estimating function $\lambda'\hat{\beta}$ is unique and the variance of the estimate is given by $V(\lambda'\hat{\beta}) = \lambda'S^-\lambda\sigma^2$, where $\hat{\beta} = S^-X'Y$ is a solution of the normal equations $S\hat{\beta} = X'Y$, S^- is a g-inverse of S, and X is not of full rank.*

In the context of singular weighing designs, a third form of a g-inverse (not mentioned in Rao [8]) is suggested here. This third form is obtained by a slight variation of the method as followed in the case of the second g-inverse. Let $A =$

$X'X = S$, where X is of dimensions $N \times p$ and rank $r < p$, and let it be possible to arrange the columns of X in such a manner that the r independent columns take the first r positions. Then a g-inverse may be defined by $S^- = P'\Delta_S^- P$, where $PSP' = \Delta_S$, and P and P' have similar operational significance as in the second g-inverse, and where

$$(2.2) \quad \Delta_S = \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta_S^- = \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \qquad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

It can be easily shown that for this g-inverse, $SS^-S = S$, $S^-SS^- = S^-$ and that (S^-S) is idempotent. The property of symmetry is also retained. In terms of the symbols used in the other two g-inverses, $X'X = S \equiv A$, $S^- \equiv A^-$ and $S_{11} \equiv A_{11}$, where A is partitioned as S. The second and third g-inverses will be denoted in this paper respectively by S_2^- and S_3^- for a distinction.

LEMMA 2.2. *If B is the first g-inverse of S, then $H = BS = S_2^-S = S_3^-S$, where $S = X'X$, and X is of dimensions $N \times p$ and rank $r < p$.*

PROOF. Let

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix},$$

where rank $(S_{11}) = r$. Consider the matrix,

$$B = \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & I_{p-r} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ P_{21} & I_{p-r} \end{bmatrix} = \begin{bmatrix} S_{11}^{-1} & 0 \\ P_{21} & I_{p-r} \end{bmatrix},$$

where $P'_{21} = -S_{11}^{-1}S_{12}$. Then, it is easy to show that

$$H = BS = \begin{bmatrix} I_r & S_{11}^{-1}S_{12} \\ 0 & 0 \end{bmatrix}.$$

B is a g-inverse of S of the 1st kind. We show now that a g-inverse of the 3rd kind S_3^- satisfies $H = S_3^-S$. Let

$$P = \begin{bmatrix} I_r & 0 \\ P_{21} & I_{p-r} \end{bmatrix}.$$

According to the above definition of P_{21},

$$PSP' = \begin{bmatrix} S_{11} & 0 \\ 0 & 0 \end{bmatrix};$$

and

$$S_3^- = P' \begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} P.$$

Hence,

$$S_3^-S = P'\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right](P')^{-1} = \left[\begin{array}{c|c} I_r & -P_{21}' \\ \hline 0 & 0 \end{array}\right] = H$$

Finally, define

$$P_* = \left[\begin{array}{c|c} P_{11} & 0 \\ \hline 0 & I_{p-r} \end{array}\right]\left[\begin{array}{c|c} I_r & 0 \\ \hline P_{21} & I_{p-r} \end{array}\right] = \left[\begin{array}{c|c} P_{11} & 0 \\ \hline P_{21} & I_{p-r} \end{array}\right],$$

where P_{11} is non-singular, satisfying $P_{11}S_{11}P_{11}' = D_r$. Let

$$S_2^- = P_*'\left[\begin{array}{c|c} D_r^{-1} & 0 \\ \hline 0 & 0 \end{array}\right](P_*')^{-1}.$$

Then, the g-inverse of the 2nd kind satisfies,

$$\begin{aligned} S_2^-S &= P_*'\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right](P_*')^{-1} \\ &= \left[\begin{array}{c|c} P_{11}' & P_{21}' \\ \hline 0 & I_{p-r} \end{array}\right]\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} (P_{11}')^{-1} & -(P_{11}')^{-1}P_{21}' \\ \hline 0 & I_{p-r} \end{array}\right] \\ &= \left[\begin{array}{c|c} I_r & -P_{21}' \\ \hline 0 & 0 \end{array}\right] = H. \end{aligned}$$

LEMMA 2.3. *$S_2^- = \Delta_s^-$, and $S_3^- = \Delta_s^-$.*

PROOF. We have

$$\begin{aligned} S_3^- = P'\Delta_s^-P &= P'\left[\begin{array}{c|c} S_{11}^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]P \\ &= \left[\begin{array}{c|c} I_r & P_{21}' \\ \hline 0 & I_{p-r} \end{array}\right]\left[\begin{array}{c|c} S_{11}^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & 0 \\ \hline P_{21} & I_{p-r} \end{array}\right] \\ &= \Delta_s^-. \end{aligned}$$

And, S_2^- is given by

$$\begin{aligned} S_2^- &= P_*'\Delta^-P_* \\ &= \left[\begin{array}{c|c} I_r & P_{21}' \\ \hline 0 & I_{p-r} \end{array}\right]\left[\begin{array}{c|c} P_{11}' & 0 \\ \hline 0 & I_{p-r} \end{array}\right]\left[\begin{array}{c|c} D_r^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} P_{11} & 0 \\ \hline 0 & I_{p-r} \end{array}\right]\left[\begin{array}{c|c} I_r & 0 \\ \hline P_{21} & I_{p-r} \end{array}\right] \\ &= \left[\begin{array}{c|c} I_r & P_{21}' \\ \hline 0 & I_{p-r} \end{array}\right]\left[\begin{array}{c|c} P_{11}'D_r^{-1}P_{11} & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & 0 \\ \hline P_{21} & I_{p-r} \end{array}\right] \\ &= \Delta_s^-, \quad \text{since} \quad P_{11}'D_r^{-1}P_{11} = P_{11}'P_{11}'^{-1}S_{11}^{-1}P_{11}^{-1}P_{11} = S_{11}^{-1}. \end{aligned}$$

Rao [8] has shown with reference to the first g-inverse that $\lambda'\hat{\beta}$ is unique for all

$\hat{\beta}$ satisfying $S\hat{\beta} = X'Y$, if $\lambda'H = \lambda'$. It is otherwise well known, as pointed out before, that, when X is not of full rank, an estimable linear function $\lambda'\beta$ has a unique estimate, if and only if there exists a solution for r in the equations $Sr = \lambda$. We now show that these two conditions are consistent.

LEMMA 2.4. *The system of equations* $Sr = \lambda$ *is consistent* (*has a solution*, r), *if and only if*, $\lambda'H = \lambda'$.

PROOF. Suppose $Sr = \lambda$ is consistent. Then, $\lambda'H = r'SH = r'SS^{-}S = r'S = \lambda'$. On the other hand, if $\lambda'H = \lambda'$, then $\lambda' = \lambda'S^{-}S = (S^{-}\lambda)'S$. Define $S^{-}\lambda = r$. Then, $Sr = \lambda$.

3. Singular weighing designs. It was pointed out in [1] that a weighing design which is optimum for determination of the weight of each individual object may not be the best for determination of the total weight; and, we may often be interested in ascertaining the total weight. For example, we may like to know the total weight of an object which has accidentally broken into pieces. A singular weighing design does not furnish under non-randomized procedures (see Zacks [9]) unbiased estimates of the individual weights, but it may furnish the estimate of the total weight; and it may so happen that such a singular design may even be better than an optimum non-singular weighing design which is usually sought for in weighing operations. The example of the singular design which has been optimized by Raghavarao [7] by the addition of one row is a case in instance. The example is

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}. \tag{3.1}$$

The Design Matrix (3.1) is of rank 2, and is therefore singular. $[X'X]$ is given by

$$[X'X] = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 4 & 2 \\ 2 & 2 & 2 \end{bmatrix} = S. \tag{3.2}$$

To reduce (3.2) to the form of Δ_s (in this case, the same as Δ), we need to premultiply it by a P and postmultiply by a P'. P, S_3^{-} and H are given below:

$$P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I_2} & 0 \\ \hline -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right], \quad S_3^{-} = \left[\begin{array}{cc|c} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \quad H = \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \hline 0 & 0 & 0 \end{array}\right].$$

It is noticed that if $\lambda' = (1, 1, 1)$, it is possible to have a unique estimate (Lemma 2.4) for $\lambda'\beta$, as we have $\lambda'H = \lambda'$. Thus, although the design is singular, it is possible to have an estimate of the total weight which may, at times, be needed in practice. The variance of the total weight will, by Lemma 2.1, be given by $V(\lambda'\hat{\beta}) = \lambda' S_3^- \lambda \sigma^2$ which is the sum of all the elements of $\sigma^2 S_3^-$, as $\lambda' = [1, 1, 1]$. Thus, the variance of the total weight is $\frac{1}{2}\sigma^2$. Now, the best chemical balance non-singular design for determining the weights of three objects from four weighing operations is given by three columns of a 4×4 Hadamard matrix, and the variance of the sum of weights for such a design would be given by $\frac{3}{4}\sigma^2$. The singular design under discussion would therefore be better than the best non-singular weighing design, when it is required to have an estimate of the total weight.

If it is desired to have an estimate of a linear function of the weights (e.g., the total weight), we may institute a comparison of the merits of two singular weighing designs on the basis of the variance of the estimate. In this context, we may think of the following singular weighing design, which has been obtained by a slight variation of the Example (3.1),

$$X = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}. \tag{3.3}$$

For the design in Example (3.3), P, S_3^-, and H are given by

$$P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I_2} & 0 \\ \hline -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right], \qquad S_3^- = \left[\begin{array}{cc|c} \frac{1}{3} & \frac{1}{6} & 0 \\ \frac{1}{6} & \frac{1}{3} & 0 \\ \hline 0 & 0 & 0 \end{array}\right], \qquad H = \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \hline 0 & 0 & 0 \end{array}\right].$$

Here also, the total weight is estimable, and the variance of the estimate is σ^2. The Singular Weighing Design (3.3), if used as one to determine total weight, will thus be worse than both the Singular Design (3.1) and the optimum design for 3 objects from 4 weighing operations, as referred to above. Comparison of efficiencies between two singular weighing designs can thus be made from an examination of the elements of S_{11}^{-1}.

4. Singular spring balance designs. Raghavarao [7] visualized three different situations under which singularity of a weighing design could perhaps occur. In fact, in spite of very good intentions on the part of an experimenter, a singular weighing design might be encountered as will be evident from the following context. D_6 furnished by Mood [5] for the case $N = p = 6$ minimizing $|X'X|$ was recognized [2] to be a Partially Balanced Incomplete Block (PBIB) design with the parameters:

$$v = b = 6, \quad r = k = 3, \quad \lambda_1 = 1, \quad n_1 = 4, \quad \lambda_2 = 2, \quad n_2 = 1,$$

$$p_{ij}^1 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \qquad p_{ij}^2 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

The above PBIB design would furnish, as pointed out by Mood [5], the most efficient non-singular spring balance design. This, however, does not mean that all PBIB designs (when $b \geqq v$) would furnish non-singular weighing designs, although all Balanced Incomplete Block (BIB) designs, as a general rule, would furnish efficient non-singular weighing designs ($b \geqq v$). As some PBIB designs furnish the most efficient spring balance designs [5], one might perhaps inadvertently adopt, with the best of intentions, any PBIB arrangement for a spring balance design. We consider in this context the following example, where $N = p = 9$. The design is furnished [3] by the blocks, (1, 2, 3), (4, 5, 6), (7, 8, 9), (1, 7, 5), (2, 9, 6), (1, 8, 6), (2, 7, 4), (3, 9, 5), (3, 8, 4) with the parameters as

$$v = b = 9, \quad r = k = 3, \quad \lambda_1 = 1, \quad n_1 = 6, \quad \lambda_2 = 0, \quad n_2 = 2,$$

$$p_{ij}^1 = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}, \quad p_{ij}^2 = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix},$$

where the 9 varieties are denoted by the 9 numbers. The above is an example of a PBIB design for $m = 2$ (i.e. two associate classes). For such a class of designs, in general, when $P_{22}^1 = 0$, it is possible to group the varieties (in weighing designs, the objects to be weighed) in such a manner that the varieties in a group are only second associates. We will arrange the varieties (1, 2, 3, 4, 5, 6, 7, 8, 9) in the order as (1, 4, 9, 2, 8, 5, 3, 7, 6), dividing the second associates into three groups.

In BIB and PBIB designs, v and b are used to denote respectively the number of varieties and the number of blocks. In weighing designs, v takes the place of the number of objects to be weighed and b that of N, the number of weighings than can be made. With this representation, X of the weighing design in the above example would take the following form:

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}. \tag{4.1}$$

The arrangement is made in the above manner to secure internal symmetry in $[X'X]$. $[X'X]$ will have, in the diagonal, 3 diagonal matrices with 3 in the diagonal and 0 as the off-diagonal elements. The other elements of $[X'X]$ will be 1.

Design (4.1) is obviously a singular one. The rank is 7. It was pointed out [2] that in the class of PBIB designs where the varieties are divisible into groups as above, a condition that gives 0 value to the det. $|X'X|$ is that $r = (1 + n_2)\lambda_1$. This condition is satisfied. If we add the columns by three's and subtract the sum of the first three from that of the second three and the third, we shall get 2 columns of zeros. We may, therefore, take the sixth and the ninth columns as dependent, and the remaining 7 columns as independent. To make up the deficiency of full rank, we would, therefore, add two more weighing operations maximizing the resultant determinant $|X'X|$, as per requirement of Mood's efficiency definition [5].

5. Maximization of the resultant determinant. In order to consider the question of maximizing the resultant determinant, we propose to build up the algebraic steps in terms of the symbols used in Section (2) and we wish to proceed as follows: We augment the design matrix X, which is of dimensions $N \times p$ and of rank $r(r < p)$, by adding to it $(p - r)$ independent rows. Let A denote the matrix of these independent rows. We arrange the columns of A in such a manner that the $(p - r)$ independent columns occupy the last $(p - r)$ positions. Then, we partition $X \equiv [X_1 \vdots X_2]$ and $A \equiv [A_1 \vdots A_2]$, where X_1 denotes the first r independent columns of X, and A_2 the last $(p - r)$ independent columns of A. Denoting the augmented matrix by

$$F = \left[\begin{array}{c} X \\ \hdashline A \end{array}\right] = \left[\begin{array}{c:c} X_1 & X_2 \\ \hdashline A_1 & A_2 \end{array}\right],$$

we have $F'F$ as given by

$$F'F = \left[\begin{array}{c:c} S_{11} & S_{12} \\ \hdashline S_{21} & S_{22} \end{array}\right] + \left[\begin{array}{c:c} A_{11} & A_{12} \\ \hdashline A_{21} & A_{22} \end{array}\right] = [S + T], \tag{5.1}$$

where $S_{11} = X_1'X_1$, $S_{12} = X_1'X_2$, $S_{21} = X_2'X_1$, $S_{22} = X_2'X_2$, $A_{11} = A_1'A_1$, $A_{12} = A_1'A_2$, $A_{21} = A_2'A_1$, $A_{22} = A_2'A_2$, and S and T are the two matrices consisting respectively of the elements S_{ij} and $A_{ij}(i, j = 1, 2)$. If we premultiply the Matrix (5.1) by P and postmultiply by P', where P and P' are as defined in Section (2), we would get S reduced to Δ_S, T to U and the whole right hand side of (5.1) to

$$\Delta_S + U = \left[\begin{array}{c:c} S_{11} & 0 \\ \hdashline 0 & 0 \end{array}\right] + \left[\begin{array}{c:c} U_{11} & U_{12} \\ \hdashline U_{21} & U_{22} \end{array}\right], \tag{5.2}$$

where

$$U_{11} = A_{11} = A_1'A_1, \qquad U_{12} = A_{11}P_{21}' + A_{12} = A_1'[A_2 + A_1P_{21}'],$$

$$U_{22} = P_{21}A_{11}P_{21}' + A_{21}P_{21}' + P_{21}A_{12} + A_{22} = [A_2 + A_1P_{21}']'[A_2 + A_1P_{21}],$$

and U_{21} as the transpose of U_{12}. It is to be noted here that A_1 and A_2 are such that $U_{12} \neq 0$. We now premultiply and postmultiply (5.2) by R and R', where R is a reduced unit matrix and will be of the form as given by

$$R = \left[\begin{array}{c:c} I_r & R_{12} \\ \hdashline 0 & I_{p-r} \end{array}\right],$$

where R_{12} has a similar meaning as P_{12} in P of Section (2). These multiplications will not alter Δ_S and U_{22}. Remembering that the determinant of a reduced unit matrix is unity, $|F'F|$ would reduce to

$$|F'F| = |S_{11}|\,|Y'Y|,$$

where Y is given by $[A_2 + A_1P'_{21}]$. Thus, given X, maximization of $|F'F|$, which we would need as per Mood's efficiency definition [5], would depend on the maximization of $|Y'Y|$. Freedom of choice of elements of Y is thus further restricted by the fact that Y involves elements of the given matrix X. [We use the expression "further" here because in weighing designs we choose the elements x_{ij} of X so as to maximize the det. $|X'X|$ as per Mood's efficiency definition [5]. The restriction imposed in the choice is that the elements have to be either 1, -1, 0 in a chemical balance, or 1, 0 in a spring balance.] A reference to equation (2.4) would reveal that $P'_{21} = -J$ of Raghavarao [7]. The value of the det. $|F'F|$ would reduce to $|X_1'X_1|\,|[A_2 - A_1J]'[A_2 - A_1J]|$ in terms of the symbol J used in [7]. One might perhaps prefer P'_{21} to the form $J = S_{11}^{-1}S_{12}$, as in many situations it may be relatively much simpler, as we shall presently see, to find P'_{21} rather than to evalute $S_{11}^{-1}S_{12}$.

If the last $(p - r)$ independent columns were allowed to take the first $(p - r)$ positions and the entire operation connected with the method of "sweep out" repeated, the factor $|S_{11}|$ of $|F'F|$ would be have been replaced by the determinant of A_{22} which is contained in $U_{22} = Y'Y$. Thus, it would be clear that to secure maximization of $|F'F|$ we would need also to maximize $|A_{22}|$.

For the example of Singular Design (3.1), $P'_{21} = [-\frac{1}{2}, -\frac{1}{2}]'$. For $|A_{22}|$ to be maximum, we may take $A_2 = -1$ or $+1$. If it is -1, A_1 is easily seen to be [1, 1]. If, however, it is $+1$, A_1 will be $[-1, -1]$. Both these rows maximize $|Y'Y|$. In fact, these two rows are virtually the same, as we can always change the signs of an entire row without affecting the design. That A_2 could also be equal to $+1$ does not appear to have been recognized in [7].

If it is not required to augment the design matrix X by the addition of a large number of rows, one can easily find by inspection the elements of $A = [A_1, A_2]$ to maximize $|Y'Y|$.

Design (3.1) is such that the third column is half the sum of the elements in the first two columns. Let us add to it a fourth column which is half the difference of the elements in the first two columns. The resultant design matrix will be of dimensions 4×4, and of rank 2. We may therefore add two rows to make the resultant design matrix of full rank, maximizing the determinant $|Y'Y|$. For this design, P'_{21}, S_{11}^{-1}, H, A_1 and A_2 are given by

$$P'_{21} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & +\frac{1}{2} \end{bmatrix}, \quad S_{11}^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad H = \left[\begin{array}{c|cc} I_2 & \frac{1}{2} & \frac{1}{2} \\ & \frac{1}{2} & -\frac{1}{2} \\ \hline 0 & \multicolumn{2}{c}{0} \end{array}\right],$$

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

The value of $|F'F| = |S_{11}|\,|Y'Y| = 4^2 \cdot 5^2$.

From H, we can find what linear combinations are estimable. In this case, the sum of the weights is not estimable, as we see that $\lambda' H \neq \lambda'$, with $\lambda' = [1, 1, 1, 1]$.

Instead of multiplying examples for the chemical balance, we would consider the problem of augmentation with reference to the example of PBIB design as in (4.1). Rank of the matrix is 7, as pointed out before, and we need to add only two rows. The sixth and the ninth columns may be shifted to the outermost positions.

We explained before what would be the constitution of the 9×9 matrix $S = X'X$. Although there is symmetry in S, it would need quite an effort to evaluate $S_{11}^{-1}S_{12}$, where S_{11} is of dimensions 7×7. We would therefore try to find P'_{21}. It can be very easily seen that P_{21} will be of the form as given by

$$P_{21} = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 1 & 1 & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & 1 & 1 \end{bmatrix}. \tag{5.3}$$

It may be recalled here that the elements of X for a spring balance design can only be 1 or 0. Hence, a maximum $|A_{22}|$ can only be given by an A_2, if A_2 is of the following form:

$$A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{or,} \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

By inspection of A_2 in conjunction with $A_1P'_{21}$, we can see that we should take the first form of A_2, and A_1 maximizing $|Y'Y|$ will then be given by

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

so that the two rows to be added would be

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

The above two rows are given by the blocks with varieties (2, 8, 5) and (3, 7, 6). Addition of the above two rows would make the value of the det. $|Y'Y| = 3^4$, while the value of $|S_{11}| = 3^5$. Thus, the value of maximized $|F'F| = 3^9$. It may be remarked here that if we add the blocks (1, 4, 9), (2, 8, 5) and (3, 7, 6) to X, we get a BIB design with $v = 9$, $b = 12$, $r = 4$, $k = 3$ and $\lambda = 1$, and it is known that BIB designs in general furnish [1, 5] optimum spring balance designs. In case of this BIB design, the value of the corresponding det. $|X'X| = 3^9 \cdot 4$. Thus, increase of weighing operations to 12 from 11 as obtained by the addition of two rows to a 9×9 singular design increases the value of the det. $|X'X|$ four fold.

It is interesting to note here that the two rows which were added through the principle of maximization of the determinant to the singular design are given by the last two of the three blocks which bring up the PBIB design (singular) to a BIB design (non-singular).

H for this singular design can be obtained by bordering an I_7 on the right by $= P'_{21}$ obtained from (5.3), and below by $0'$s. From H, and the relationship $\lambda' H = \lambda'$ we can easily find (Lemma 2.4) what linear functions will be estimable. It will be noticed that the sum is not estimable in this case, but a function such as $\lambda'\beta$, with $\lambda' = [1, 1, 1, 1, 1, 1, 1, -1, -1]$, is estimable. On the basis of the variance of this linear estimate, the PBIB design under discussion may be compared with the best available BIB design with the parameters, $v = 9$, $b = 12$, $r = 4$, $n = 3$ and $\lambda = 1$, as referred to before. To calculate the variance under the PBIB design, we would need to know S_{11}^{-1} occurring in Δ_S^-. S_{11}^{-1} is shown below:

$$S_{11}^{-1} = \begin{bmatrix} \frac{7}{9} & \frac{4}{9} & \frac{4}{9} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{4}{9} & \frac{7}{9} & \frac{4}{9} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{4}{9} & \frac{4}{9} & \frac{7}{9} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

The variance of the above mentioned linear function, as estimated from the PBIB design, will be the sum of the elements of S_{11}^{-1} multiplied by σ^2. The sum of the elements is unity. Hence, the variance is σ^2. This design has 9 weighing operations. For the above BIB design with twelve weighing operations which is the best available spring balance design for individual weights for the given parameters, the diagonal elements of the corresponding $[X'X]^{-1}$ will be $\frac{11}{36}$, and the off-diagonal elements $-\frac{1}{36}$. Hence, the variance of the above linear function will be $(\frac{83}{36})\sigma^2$ which is larger than σ^2. Thus, the PBIB design under discussion may be preferred for the estimation of a linear function such as this or similar functions.

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