EXACT POWER OF MANN-WHITNEY TEST FOR EXPONENTIAL AND RECTANGULAR ALTERNATIVES[1]

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0. Summary. A closed form expression for the exact distribution of the Mann-Whitney-Wilcoxon U test has been derived. From this, expressions for the exact power of the U test for exponential and rectangular alternatives have been derived. Several determinations of the power of combined sample sizes of 11, 15 and 21 have been compared with the corresponding determinations of the power of Mood's median test procedure. Also, the asymptotic efficiency of the U test relative to Mood's median test for exponential and rectangular translation alternatives is considered.

1. Introduction. Mann and Whitney [10] have derived a rank test which is a modification of an earlier test proposed by Wilcoxon [17] for distinguishing between two populations. van der Vaart [16] has derived a closed form expression for certain probabilities of Wilcoxon's two sample test under the null hypothesis. Its asymptotic normality was established by Lehmann [7]. The power properties of the Mann-Whitney-Wilcoxon U test for some parametric alternatives have been considered by Lehmann [8], Sundrum [12], van der Vaart [14], [15], van Dantzig [13], Dixon [2], Dwass [3] and others. Asymptotic relative efficiency of the U test has been considered by Mood [11], Hodges and Lehmann [4], [5] and Witting [18].

The purpose of our investigation is: (i) to derive a closed form expression for the distribution of the U test for the null and the non-null hypotheses; (ii) to derive the expression for the exact power under exponential and rectangular alternatives; (iii) to compare the small sample exact power of the U test with exact power of Mood's [11] median test for exponential and rectangular alternatives; and (iv) to study the asymptotic efficiency of the U test relative to the Mood's median test with exponential and rectangular alternatives.

2. Mann-Whitney U test. Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be independently distributed with continuous cumulative distribution functions (cdfs) F and G respectively. We want to test the hypothesis

$$H_0 : F(x) = G(x),$$

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against the alternative H_1 given by

$$H_1 : F(x) > G(x).$$

Let $N = m + n$ denote the size of the combined sample and $Z_{(1)} < Z_{(2)} < \cdots < Z_{(N)}$ be the combined ordered X's and Y's. This ordering is unique with probability 1, due to the assumption of continuity of F and G.

Mann and Whitney [10] defined a statistic U which is equal to the number of times a Y precedes an X in the combined ordered sample. Then, a test of size α based on the Mann-Whitney U statistic is:

$$\text{reject } H_0 \text{ if } U \leqq u_\alpha \text{ and}$$
$$\text{accept } H_0 \text{ if } U > u_\alpha, \quad \text{where } \Pr\{U \leqq u_\alpha \mid H_0\} \leqq \alpha.$$

This U statistic is related to Wilcoxon's T statistic (the sum of the ranks of X's) by

$$U = mn + \tfrac{1}{2}n(n + 1) - T, \tag{2.1}$$

which gives a simple way of computing U from the observed value of T. The exact distribution of U under the null hypothesis H_0 has been tabulated by Mann and Whitney [10].

3. The null distribution of the U test. Mann and Whitney have shown that the null distribution can be calculated recursively from

$$P_{m,n}(u) = [m/(m + n)]P_{m-1,n}(u - n) + [n/(m + n)]P_{m,n-1}(u), \tag{3.1}$$

with

$$\begin{aligned} P_{0,n}(u) &= 0 && \text{if } u > 0; \\ P_{m,0}(u) &= 0 && \text{if } u > 0; \\ P_{0,n}(u) &= 1 && \text{if } u = 0; \\ P_{m,0}(u) &= 1 && \text{if } u = 0; \\ P_{m,n}(u) &= 0 && \text{if } u < 0; \end{aligned}$$

where $P_{m,n}(u) = \Pr\{U = u \mid H_0\}$ for samples of size m and n. van der Vaart [16] has derived closed form expressions for $P(U = u \mid H_0)$ and $P(U \leqq u \mid H_0)$ in the form of determinants. For large U, the calculations may be somewhat tedious. However, it is possible to express the null distribution in closed form and to simultaneously derive a joint density function which can be used to calculate the exact power under fixed alternatives.

Let us first consider that the set $\{y_i \mid i = 1, 2, \cdots, n\}$ has been chosen from G noting that there are n factorial ways of obtaining the set. Next, we order the set and then compute the probability of choosing a set $\{x_j \mid j = 1, 2, \cdots, m\}$ from F such that a specific value for U is obtained. (That is, we want an expression for the joint distribution of U and the Y's.)

Any such set of m values of x from F will be spaced between the n previously chosen y values in the following manner: Let i_k denote the number of x values between y_k and y_{k+1} $(k = 1, 2, \cdots, n-1)$, i_0 the number of x values less than y_1 and i_n the number of x values greater than y_n as displayed in Figure 1.

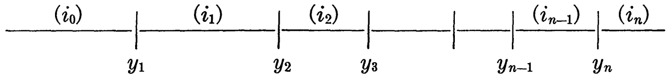

FIG. 1

Any particular arrangement can be denoted by the $n + 1$ dimensional integer vector $\mathbf{i} = (i_0, i_1, i_2, \cdots, i_n)$. A vector $\mathbf{i}$ is said to be admissible if

$$(3.2)\qquad \begin{aligned} &\text{(i)} && i_k \geqq 0 \qquad (k = 0, 1, \cdots, n);\\ &\text{(ii)} && \textstyle\sum_{k=0}^{n} i_k = m;\\ &\text{(iii)} && \textstyle\sum_{k=0}^{n} k i_k = u. \end{aligned}$$

For convenience, let I be the set of all admissible vectors $\mathbf{i}$. Now consider the probability of choosing i_0 values of x less than y_1 say P_0, i_1 values of x in the interval $y_2 - y_1$ say P_1, and so on until the i_n values of x greater than y_n say P_n. These probabilities are given by:

$$(3.3)\qquad \begin{aligned} P_1 &= [m!/i_1!(m - i_1)!](F_2 - F_1)^{i_1};\\ P_2 &= [(m - i_1)!/i_2!(m - i_1 - i_2)!](F_3 - F_2)^{i_2};\\ P_n &= [(m - \textstyle\sum_{k=1}^{n-1} i_k)!/i_n!(m - \textstyle\sum_{k=1}^{n} i_k)!](1 - F_n)^{i_n}\\ &= [(m - \textstyle\sum_{k=1}^{n-1} i_k)!/i_n!\, i_0!](1 - F_n)^{i_n};\\ P_0 &= F_1^{i_0}; \end{aligned}$$

where $F_i = F(y_i)$, $i = 1, 2, \cdots, n$.

The resulting joint density function of $u, y_1, y_2, \cdots, y_n$ is given by the product of the above probabilities and the probability of choosing the n values of y summed up over all admissible values of the vector $\mathbf{i}$ as follows:

$$(3.4)\qquad \begin{aligned} h(u, y_1, \cdots, y_n) = m!\,n! \textstyle\sum_I [\prod_{k=0}^{n} i_k!]^{-1} F_i^{i_0}(F_2 - F_1)^{i_1} \cdots\\ \cdot(1 - F_n)^{i_n}(dG_1/dy_1)(dG_2/dy_2) \cdots (dG_n/dy_n), \end{aligned}$$

where $G_i = G(y_i)$, $i = 1, 2, \cdots, n$.

Now the distribution of u under the null hypothesis: $F = G$, can be found by integrating the y's over the range $-\infty < y_1 < \cdots < y_n < \infty$. To simplify the integration, the variables of integration are transformed from y_i to $F(y_i) = F_i$, and the new range of integration is $0 < F_1 < \cdots < F_n < 1$. Substituting $F_i = G_i$, $i = 1, 2, \cdots, n$, into (3.4) and integrating yields

$$(3.5)\qquad \begin{aligned} \varphi_0(u) = m!\,n! \textstyle\sum_I [\prod_{k=0}^{n} i_k!]^{-1} \int_0^1 \int_0^{F_n} \cdots \int_0^{F_2} F_1^{i_0}(F_2 - F_1)^{i_1} \cdots\\ \cdot(1 - F_n)^{i_n}\, dF_1\, dF_2 \cdots dF_n, \end{aligned}$$

which integrates and simplifies to

$$\begin{aligned}(3.6)\qquad \varphi_0(u) &= [m!\,n!/(m+n)!]\textstyle\sum_I 1\\ &= [m!\,n!/(m+n)!]\cdot[\text{number of elements in the set } I].\end{aligned}$$

4. Power of U test against the alternatives of translation in the exponential population. Here, the alternative hypothesis considered is

$$(4.1)\qquad H_a: \quad \begin{aligned} F(x) &= 1 - e^{-x}, && x \geqq 0,\\ &= 0, && x < 0,\\ G(y) &= 1 - e^{-(y-a)}, && y \geqq a,\\ &= 0, && y < a, \quad \text{where} \quad a > 0.\end{aligned}$$

Let $\varphi_a(u)$ denote the probability of U taking on the value u given that H_a is true. Then

$$(4.2)\qquad \varphi_a(u) = \int \cdots \int h(u, y_1, y_2, \cdots, y_n)\, dy_1 \cdots dy_n,$$

where h is given by (3.4). For convenience of notation, let

$$(4.3)\qquad \eta = e^{-a}, \qquad \gamma = 1 - \eta.$$

Then

$$(4.4)\qquad F_j = \gamma + \eta G_j = 1 - \eta(1 - G_j) \qquad \text{for} \qquad a \leqq y_j < \infty.$$

Substituting (4.3) and (4.4) into (4.2) and expanding the term $(\gamma + \eta G_1)^{i_0}$ yields

$$(4.5)\qquad \varphi_a(u) = m!\,n! \textstyle\sum_I \sum_{v=0}^{i_0} [\prod_{k=0}^{n} i_k\,!]^{-1}\binom{i_0}{v}\gamma^{i_0-v}\eta^{m-i_0+v}\int_0^1\int_0^{G_n}\cdots \int_0^{G_2} G_1^{\,v}(G_2 - G_1)^{i_1}\cdots(1-G_n)^{i_n}\,dG_1\,dG_2\cdots dG_n.$$

Transforming the variables of integration and integrating gives

$$(4.6)\qquad \varphi_a(u) = m!\,n! \textstyle\sum_I \sum_{v=0}^{i_0} \gamma^{i_0-v}\eta^{m-i_0+v}[(i_0 - v)!(m + n - i_0 + v)!]^{-1}.$$

The power of the test can be computed from (4.6) by evaluating

$$\Pr\{U \leqq u_\alpha \mid H_a\} = \textstyle\sum_{u=0}^{u_\alpha} \varphi_a(u).$$

5. Power of U test against the alternatives of change in location and scale in the rectangular population. The alternative hypothesis is given by

$$\begin{aligned} F(x) &= x, && 0 \leqq x \leqq 1,\\ &= 0, && x < 0,\\ &= 1, && x > 1,\end{aligned}$$

(5.1) $H_{a\theta}$:

$$G(y) = (y - a)/\theta, \qquad a \leqq y \leqq a + \theta,$$
$$= 0, \qquad y < a,$$
$$= 1, \qquad y > a + \theta, \quad \text{where} \quad a > 0, \theta > 0.$$

Let $\varphi_{a\theta}(u)$ denote the probability of U taking on the value u given that $H_{a\theta}$ is true. Then, as in (4.2),

$$\varphi_{a\theta}(u) = \int \cdots \int h(u, y_1, \cdots, y_n)\, dy_1 \cdots dy_n, \tag{5.2}$$

where h is given by (3.4). For notational convenience, let

$$b = 1 - a. \tag{5.3}$$

There are two cases to be distinguished, namely:

(i) $a + \theta \leqq 1$, (ii) $a + \theta > 1$.

5.1 $a + \theta \leqq 1$. A development similar to that used in Section 4 yields

$$\varphi_{a\theta}(u) = m!\, n! \sum_I \sum_{v=0}^{i_0} \sum_{q=0}^{i_n} (-1)^q a^{i_0-v} b^{i_n-q} G^{m-i_0-i_n+v+q}[q!(i_0 - v)! \cdot (i_n - q)!\,(m + n - i_0 - i_n + v - 1)!\,(m + n - i_0 - i_n + v + q)]^{-1}. \tag{5.4}$$

5.2 $a + \theta = 1$. If we define $\eta = 1 - a'$, $\gamma = 1 - \eta$ this case is identical to that in Section 4 with $e^{-a} = 1 - a'$.

5.3 $a + \theta > 1$. This case can be further subdivided into two subcases, namely:

(1) $a < 1$, (ii) $a \geqq 1$.

5.3.1 $a < 1$. For $a < 1$, the range of integration for y can be split into four parts, namely: (1) $-\infty < y < a$; (2) $a \leqq y \leqq 1$; (3) $1 < y \leqq a + \theta$; (4) $a + \theta < y < \infty$. Over parts (1) and (4) the value of the integral is zero since G is constant. Hence, we will consider only the ranges (2) and (3).

For notational convenience, let I_j be the subset of I such that $i_k = 0$ for $j < k \leqq n$; then define

$$P_j(u) = \sum_{I_j} \sum_{v=0}^{i_0} \sum_{q=0}^{i_j} [(-1)^q a^{i_0-v} b^{m-i_0+v+j}][\theta^j q!(i_0 - v)!(i_j - q)! \cdot (m - i_0 - i_j + v + j - 1)!(m - i_0 - i_j + v + q + j)]^{-1} \tag{5.5}$$

for $j \geqq 1$ and

$$P_0(u) = \delta_{u,0}/m! \qquad \text{where} \qquad \delta_{u,0} = 0 \quad \text{if } u \neq 0$$
$$= 1 \quad \text{if } u = 0.$$

Substituting these results into (5.2) and breaking up the range of integration properly yields

$$\varphi_{a\theta}(u) = m!\,n! \sum_{j=0}^{n} \{P_j(u)(1 - b/\theta)^{n-j}[(n - j)!]^{-1}\}. \tag{5.6}$$

5.3.2 $a \geqq 1$. In the case $a \geqq 1$, the results are trivial, namely:

$$(5.7)\quad \varphi_{a\theta}(u) = n!\,\delta_{u,0} \int_{y_n=a}^{a+\theta} \int_{y_{n-1}=a}^{y_n} \cdots \int_{y_1=a}^{y_2} dG_1\, dG_2 \cdots dG_n = \delta_{u,0}\,.$$

Using the above results for $\varphi_{a\theta}(u)$, the power of the U test under the alternative hypothesis can be calculated from

$$\Pr\{U \leqq u_\alpha \mid H_{a\theta}\} = \sum_{u=0}^{u_\alpha} \varphi_{a\theta}(u).$$

Exact power of Mann-Whitney test and Mood's median test with exponential translation alternatives for certain chosen sample sizes and levels of significance are given in Table 1.

TABLE 1

Power of Mood's median test, Mann-Whitney's U test—exponential distribution
$F(x) = 1 - e^{-x}, \quad G(y) = 1 - e^{-(y-a)}, \quad a > 0$

N	m	u_0	a=0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.5	2.0	3.0
11	4	4	.0152	.0241	.0370	.0545	.0765	.1031	.1338	.1680	.2049	.2438	.2841	.4851	.6536	.8591*
		2	.0121	.0237	.0418	.0661	.0961	.1307	.1690	.2099	.2525	.2959	.3395	.5424	.6996	.8808†
	5	5	.0022	.0039	.0071	.0126	.0211	.0334	.0499	.07(8	.0960	.1252	.1577	.3502	.5418	.8044
		0	.0022	.0039	.0071	.0126	.0211	.0334	.0499	.0708	.0960	.1252	.1577	.3502	.5418	.8044
		4	.0671	.0975	.1371	.1855	.2410	.3015	.3647	.4283	.4906	.5500	.6057	.8141	.9209	.9877
		6	.0628	.1019	.1523	.2115	.2764	.3441	.4121	.4781	.5408	.5990	.6523	.8424	.9345	.9901
	6	5	.0130	.0214	.0351	.0561	.0860	.1252	.1728	.2273	.2868	.3490	.4120	.6887	.8579	.9764
		3	.0152	.0249	.0399	.0611	.0886	.1220	.1605	.2030	.2484	.2955	.3432	.5644	.7277	.8993
	7	5	.0455	.0678	.1007	.1464	.2053	.2751	.3519	.4316	.5102	.5847	.6529	.8803	.9657	.9978
		5	.0545	.0808	.1163	.1610	.2137	.2725	.3351	.3994	.4631	.5246	.5826	.8027	.9163	.9871
15	5	5	.0070	.0121	.0200	.0314	.0467	.0663	.0900	.1178	.1491	.1833	.2200	.4175	.5979	.8320
		5	.0063	.0148	.0296	.0514	.0799	.1145	.1538	.1968	.2421	.2887	.3355	.5510	.7122	.8893
		4	.1002	.1421	.1924	.2494	.3110	.3746	.4384	.5004	.5594	.6144	.6648	.8466	.9358	.9902
		14	.1032	.1725	.2521	.3356	.4180	.4962	.5680	.6324	.6893	.7388	.7813	.9140	.9675	.9955
	7	6	.0089	.0158	.0271	.0447	.0699	.1033	.1450	.1939	.2485	.3072	.3679	.6503	.8354	.9717
		7	.0070	.0146	.0281	.0491	.0785	.1161	.1609	.2115	.2662	.3231	.3806	.6374	.8056	.9461
	7	5	.1002	.1454	.2024	.2697	.3441	.4219	.4994	.5734	.6418	.7031	.7569	.9219	.9785	.9987
		16	.0946	.1586	.2379	.3263	.4172	.5054	.5872	.6603	.7240	.7781	.8233	.9490	.9868	.9992
15	8	7	.0012	.0025	.0050	.0100	.0189	.0336	.0556	.0858	.1243	.1704	.2227	.5190	.7561	.9549
		4	.0019	.0038	.0074	.0139	.0244	.0397	.0606	.0870	.1186	.1549	.1950	.4185	.6172	.8520
		6	.0317	.0513	.0806	.1215	.1746	.2385	.3105	.3870	.4645	.5397	.6101	.8579	.9577	.9972
		12	.0361	.0648	.1072	.1631	.2300	.3043	.3820	.4594	.5336	.6025	.6649	.8726	.9564	.9952
	10	7	.0186	.0307	.0506	.0823	.1295	.1932	.2713	.3593	.4511	.5412	.6251	.8958	.9786	.9994
		8	.0200	.0328	.0525	.0810	.1196	.1681	.2253	.2892	.3571	.4263	.4945	.7686	.9095	.9882
21	7	7	.0010	.0222	.0043	.0079	.0137	.0222	.0339	.0493	.0686	.0917	.1184	.2917	.4839	.7713
		10	.0011	.0037	.0099	.0214	.0397	.0652	.0980	.1317	.1814	.2294	.2798	.5266	.7125	.9000
		6	.0209	.0359	.0579	.0879	.1261	.1720	.2242	.2812	.3410	.4020	.4623	.7190	.8722	.9788
		21	.0189	.0451	.0871	.1438	.2120	.2873	.3655	.4428	.5165	.5847	.6465	.8558	.9458	.9928
	10	8	.0073	.0143	.0266	.0467	.0768	.1183	.1713	.2342	.3046	.3792	.4548	.7694	.9244	.9946
		20	.0064	.0158	.0345	.0656	.1108	.1693	.2384	.3146	.3939	.4725	.5476	.8227	.9416	.9946
	11	9	.0016	.0035	.0074	.0149	.0285	.0506	.0833	.1276	.1831	.2480	.3194	.6734	.8837	.9909
		14	.0014	.0035	.0085	.0182	.0352	.0613	.0976	.1439	.1988	.2605	.3263	.6419	.8423	.9760
		8	.0226	.0402	.0683	.1099	.1663	.2369	.3182	.4058	.4943	.5794	.6574	.9051	.9804	.9995
		26	.0215	.0463	.0883	.1491	.2266	.3156	.4097	.5028	.5902	.6685	.7364	.9303	.9846	.9993
	14	10	.0028	.0057	.0115	.0230	.0449	.0824	.1394	.2161	.3085	.4096	.5118	.8744	.9799	.9997
		13	.0028	.0056	.0110	.0208	.0372	.0624	.0980	.1446	.2013	.2662	.3366	.6770	.8787	.9877
		9	.0426	.0697	.1113	.1713	.2511	.3477	.4539	.5608	.6600	.7460	.8164	.9753	.9978	1.0000
		26	.0469	.0840	.1401	.2160	.3081	.4096	.5124	.6095	.6959	.7691	.8286	.9699	.9959	.9999

* First row—Mood's Median Test.

† Second row—Mann-Whitney's U Test.

6. Asymptotic results. It is well known (see Lehmann [7]) that $(U - EU)/(\text{Var } U)^{\frac{1}{2}}$ tends to the standard normal variable as $m, n \to \infty$ such that $n/m \to$ constant $< \infty$, where

$$\begin{aligned} E(U) &= mn \textstyle\int G\, dF \\ &= (mn/2)e^{-a}, \quad \text{for the exponential alternatives (see (4.1))}, \\ &= mn\theta/2, \quad \text{for the rectangular alternatives (see Section 5.1, (i)}: a + \theta \leqq 1) \\ &= mnb^2/2\theta \quad \text{for (ii)}: a + \theta > 1\ (a < 1, b = 1 - a) \end{aligned}$$

and $\text{Var } U = mn\{[(m + n + 1)/12] + (m - 1)(\lambda - \epsilon_1) + (n - 1)(\lambda - \epsilon_2) - (m + n - 1)\lambda^2\}$ with $\lambda = \frac{1}{2} - \int G\, dF$, $\epsilon_1 = \frac{1}{3} - \int G^2\, dF$, $\epsilon_2 = \frac{1}{3} - \int (1 - F)^2\, dG$. The Pitman efficiency of the test procedure τ^* relative to τ is given by

$$E_{\tau^*,\tau} = \lim_{\theta \to \theta_0} \{\sigma(\theta_0)(\mu^{*\prime})(\theta_0)/\sigma^*(\theta_0)\mu'(\theta_0)\}^2,$$

where $\mu(\theta)$, $\sigma^2(\theta)(\mu^*(\theta), (\sigma^*)^2(\theta))$ are the asymptotic mean and variance of $\tau(\tau^*)$. For Mood's [11] median test

$$\mu_M(\theta) = mF(c), \qquad \sigma_M^2(\theta_0) = mn/4(m + n)$$

where c is the solution of $mF(c) + nG(c) = (m + n)/2$. Computations yield $E_{M,U} = \frac{1}{3}$ either for the exponential translation alternatives, or the rectangular translation alternatives or the rectangular translation and scale alternatives in which $G(x) = (x - a)/(1 - a)$, $a \leqq x \leqq 1$.

7. Discussion of Table 1. The exact powers of the Mann-Whitney U test for exponential alternative and for total sample sizes 11, 15 and 21 given in Table 1 are computed from Equation (4.6) by performing the indicated sum. As a check, the values for the null distribution ($a = 0$ in Table 1) were compared with direct calculations using (3.6) and the recurrence formulae of Mann and Whitney (3.1). The powers of Mood's median test are taken from Leone, et al. [9].

It should be noted that when the location parameter is zero, we get the null distribution with the power equal to the level of significance, α. Since the distributions of the test statistics are discrete, the values of α do not in general coincide for both the tests. Hence, although many different cases have been computed, only those values that are relatively close together and which indicate the general trend, have been tabulated. The conclusions that can be drawn from this table (relative to the exponential alternatives) are:

(1) If m is smaller than n, the Mann-Whitney test is more powerful than Mood's test. To note this increase of power, several cases were intentionally chosen where the level of significance for the Mann-Whitney test was slightly less than that of the Mood test. In these cases, the power of Mann-Whitney's test overtakes Mood's test as the location parameter, a, increases.

(2) If m is larger than n, Mood's test is more powerful than the Mann-Whitney test. Likewise, to note this increase of power, several cases were intentionally chosen where the level of significance for Mood's test was slightly less than that of the Mann-Whitney test. In these cases, the power of Mood's test overtakes Mann-Whitney's test as a increases.

(3) In those cases in which $m \approx n$, the two test procedures seem to exhibit powers that are approximately the same.

It is interesting to note that these conclusions based on the exact results of Table 1 are quite different from those suggested by the asymptotic efficiency discussed in Section 6.

Since rectangular alternatives for the special case in which $\theta = 1 - a'$ are related to the exponential alternatives by the simple transformation: $e^{-a} = 1 - a'$, the results in Table 1 also apply to this special case of rectangular alternatives.

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