# LIMIT THEOREMS FOR STOPPED RANDOM WALKS II[1]

By R. H. Farrell

*Cornell University*

**1. Introduction.** Throughout we suppose $\{X_n, n \geqq 1\}$ to be a stationary metrically transitive sequence of $k$-dimensional random (column) vectors such that the components of $X_1$ are all *positive* with probability one and such that $E(X_1^T X_1)^{\frac{1}{2}} < \infty$. We will use the superscript "$T$" to indicate transpose. Let $h(\cdot)$ be a real valued homogeneous function of degree one defined and continuous throughout Euclidean $k$-space. We assume that on the open first quadrant $Q = \{x \mid \min_{1 \leqq i \leqq k} x_i > 0\}$ that $h(x) > 0$ and that on $Q$, $h$ has continuous positive first partial derivatives. We will let $\alpha$ be the column vector of first partial derivatives of $h$ evaluated at $\mu$. The assumptions of this paragraph will be used throughout the remainder of the paper without further comment.

If $n \geqq 1$, define $S_n = X_1 + \cdots + X_n$, $S_0 = 0$. We use the notation $h(S_n) = H_n$, $n \geqq 0$. We define $N(t)$ to be the number of values $H_n$, $n \geqq 1$, which are less than $t \geqq 0$. Then if $t \geqq 0$, with probability one, $N(t) < \infty$ and $\lim_{t\to\infty} \cdot N(t)/t = 1/h(\mu)$. See Farrell [2]. Without risk of ambiguity we may define for $t \geqq 0$, $H_t = h(S_{N(t)})$. From our definitions it follows that with probability one, $H_t < t$ for all $t > 0$.

In this paper we are interested in studying the continuous parameter process $X(\cdot)$ defined by $X(t) = t - H_t$ if $t \geqq 0$. In the case that $\{X_n, n \geqq 1\}$ is a sequence of independently and identically distributed real valued random variables Doob [1] showed by use of Cesàro averages the construction of a stationary Markov measure (stationary under translations) for the continuous parameter process in which the joint distribution of the spacings between $m$ successive jumps is the joint distribution of $X_1, \cdots, X_m$, $m \geqq 1$. The spacing from $t = 0$ to the first jump has a different distribution which is uniquely determined by the requirement that the resulting process be stationary. This is clear from the results of Doob, op. cit.

It is the primary purpose of this paper to generalize this result for point processes $\{H_n, n \geqq 1\}$ defined above and constructed from sums of random variables having a stationary distribution. In several dimensions, unless $h$ is linear, there is no corresponding result except in the limit as $t \to \infty$. That is to say, in many ways as $n$ becomes large the two sequences of random variables, $\{H_n, n \geqq 1\}$ and $\{\alpha^T X_n, n \geqq 1\}$ look approximately the same. It is this fact of linearization that underlies the theorem stated at the end of this section. We are concerned with the construction of a stationary measure on $K_+$, defined below, such that the joint distribution of $m$ successive jumps is the joint distribution of $\alpha^T X_1, \cdots, \alpha^T X_m$, valid if $m \geqq 1$. This stationary measure is gen-

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$$\lim_{t\to\infty} t^{-1}\int_0^t P(X(s+s_1) < \gamma_1, \cdots, X(s+s_m) < \gamma_m)\, ds$$

computed for suitable values of $s_1, \cdots, s_m, \gamma_1, \cdots, \gamma_m$, $m \geqq 1$.

We now introduce more notation. Let $I(\cdot)$ be the identity function of $(-\infty, \infty)$, that is, if $-\infty < t < \infty$ then $I(t) = t$. Let $G$ be the set of all nondecreasing real valued functions defined on $(-\infty, \infty)$ such that if $g \varepsilon G$ and $t$ is a point of increase of $g$ then $g(t+) = t$. We let $K$ be the set of all real valued functions defined on $(-\infty, \infty)$ satisfying, if $k \varepsilon K$ then for some $g \varepsilon G$, $k(t) = I(t) - g(t)$ for $-\infty < t < \infty$. And we let $K_+$ be those functions which are restrictions to $[0, \infty)$ of functions in $K$.

In the sequel we think of each function $k \varepsilon K$ as being representative of an equivalence class. We say $k_1$ is equivalent to $k_2$ if at all points of continuity of the two functions their values are equal. Throughout we will be interested only in properties that are invariant under choice of representative. It is obvious that $\lim_{t\to t_0+} k(t) = k(t_0+)$ is independent of the choice of representative in an equivalence class.

In order to be precise we describe a topology on $K_+$ considered as a collection of equivalence classes. Let

$$\rho_+(k_1, k_2) = \sup_{n\geqq 1}\min\left(\int_0^n |k_1(x) - k_2(x)|\, dx, 1/n\right).$$

One may verify without too much difficulty that with this metric $K_+$ is a separable locally compact metric space in which closed bounded sets are compact. Similarly the metric

$$\rho(k_1, k_2) = \sup_{n\geqq 1}\min\left(\int_{-n}^n |k_1(x) - k_2(x)|\, dx, 1/n\right)$$

makes $K$ into a locally compact metric space in which bounded closed sets are compact, where again points are equivalence classes. Part of our argument will be based on the observation that if $\gamma > 0$ then the set $\{k \mid k(0+) \leqq \gamma\}$ is a compact subset of $K_+$ in the topology described (the above set is closed and bounded.)

The sample paths of the process $X$ are in $K_+$ and the process induces a probability measure $P$ on the Borel subsets of $K_+$. In the sequel we use $X$ as notation for a function in $K_+$.

We consider the family $\{T_t, t \geqq 0\}$ of continuous maps of $K_+$ to $K_+$ defined by,

(1.1) if $k \varepsilon K_+$ and if $t \geqq 0$, $s \geqq 0$, then

$$(T_t k)(s) = k(t+s).$$

Define a family of probability measures on the Borel subsets of $K_+$ by,

(1.2) if $t > 0$ then

$$P_t(\{X \mid X \varepsilon A\}) = t^{-1}\int_0^t P(T_s^{-1}\{X \mid X \varepsilon A\})\, ds.$$

We will be interested in showing that the set of measures $\{P_t, t > 0\}$ has a unique weak limit point not contained in the set $\{P_t, t > 0\}$. We will subsequently write $P^* = \text{weak} \lim_{t\to\infty} P_t$. In showing the existence of $P^*$ we will make use of the spacing between zeros of the sample paths, which we now define. We define a zero of the function $X \varepsilon K$ to be a number $t$ such that $X(t+) = 0$. This is a property of equivalence classes. For almost all $X$ (relative to the measure $P$) the positive zeros are a well ordered set of real numbers, this being valid also for all $P_t$, $t > 0$. We define functions $\{Z_n, n \geqq 1\}$ on $K_+$ by, $Z_0(X) =$ the time of the first zero of $X$ in $[0, \infty)$; if $n \geqq 1$ then $Z_n(X) =$ the separation of the $n + 1$st and $n$th zeros of $X$.

For later reference we define a set $A_\epsilon^m$ by

$$\text{if } m \geqq 1 \text{ and } \epsilon > 0 \quad \text{then}$$

$$(1.3) \quad A_\epsilon^m = \{X \mid \text{if } 1 \leqq i \leqq m \quad \text{then} \quad X(0+) \leqq 1/\epsilon, \\ Z_0(X) \leqq 1/\epsilon, \epsilon \leqq Z_i(X) \leqq 1/\epsilon\}.$$

If $\epsilon > 0$ then the set $A_\epsilon^m$ is closed and on this set the functions $Z_0, \cdots, Z_m$ are continuous functions of the equivalence classes of $K_+$.

THEOREM. *$P^* = \text{weak} \lim_{t\to\infty} P_t$ is a uniquely determined probability measure on the Borel subsets of $K_+$. The values of $P^*$ are determined by the values of integrals (2.3). $P^*$ is an invariant measure under the semigroup of transformations $\{T_t, t \geqq 0\}$. Under $P^*$ the joint distribution of $Z_1, \cdots, Z_m$ is the same as the joint distribution of $\alpha^T X_1, \cdots, \alpha^T X_m$.*

## 2. Lemmas and proof of the theorem.

LEMMA 2.1. *Suppose $L(\cdot)$ is a real valued measurable function of $m$ variables such that $L(\cdot)$ is a nondecreasing function of each of its variables. Let $F_m(\cdot)$ be the joint distribution function of $\alpha^T X_1, \cdots, \alpha^T X_m$. If $\epsilon > 0$ then with probability one*

$$(2.1) \quad \begin{aligned} &\int \cdots \int L((1-\epsilon)z_1, \cdots, (1-\epsilon)z_m) F_m(dz_1, \cdots, dz_m) \\ &\leqq \liminf_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^n L(H_{i+1} - H_i, \cdots, H_{i+m} - H_{i+m-1}) \\ &\leqq \limsup_{n\to\infty} n^{-1} \textstyle\sum_{i=1}^n L(H_{i+1} - H_i, \cdots, H_{i+m} - H_{i+m-1}) \\ &\leqq \int \cdots \int L((1+\epsilon)z_1, \cdots, (1+\epsilon)z_m) F_m(dz_1, \cdots, dz_m). \end{aligned}$$

PROOF. Given $\epsilon > 0$ with probability one for all large values of $n$, $(1-\epsilon)\alpha^T X_{n+1} \leqq H_{n+1} - H_n \leqq (1+\epsilon)\alpha^T X_{n+1}$. Substitution of these inequalities and use of the law of large numbers for metrically transitive stationary random variables completes the proof.

LEMMA 2.2. *Let $m \geqq 1$ be given and let $J(\cdot)$ be a bounded real valued function of $m + 1$ variables such that if $s \geqq 0$ then $J(s, \cdots, \cdot)$ is a nondecreasing function of each of the remaining $m$ variables. Suppose that $\int_0^{z_1} J(s, z_1 - s, z_2, \cdots, z_m)\, ds$*

*is a continuous function of* $z_1, \cdots, z_m$. *Let* $F_m$ *be as in Lemma* 2.1. *Then*

$$\begin{aligned}\lim_{t\to\infty} t^{-1}\int_0^t EJ(X(s+), H_{N(s)+1} - s, H_{N(s)+2} - H_{N(s)+1}, \\ \cdots, H_{N(s)+m} - H_{N(s)+m-1})\, ds \\ = (h(\mu))^{-1}\int \cdots \int [\int_0^{z_1} J(s, z_1 - s, z_2, \\ \cdots, z_m)\, ds] F_m(dz_1, \cdots, dz_m).\end{aligned} \tag{2.2}$$

PROOF. If $H_i < s \leqq H_{i+1}$ then $N(s) = i$. Therefore, using Fubini's theorem and the bounded convergence theorem, we find

$$\begin{aligned}\lim_{t\to\infty} t^{-1}\int_0^t EJ(X(s+), H_{N(s)+1} - s, \cdots, H_{N(s)+m} - H_{N(s)+m-1})\, ds \\ = E\{(\lim_{t\to\infty} N(t)/t)\lim_{n\to\infty} n^{-1} \\ \cdot \sum_{i=0}^{n} \int_0^{H_{i+1}-H_i} J(s, H_{i+1} - H_i - s, \cdots, H_{i+m} - H_{i+m-1})\, ds\} \\ = (h(\mu))^{-1}\int \cdots \int [\int_0^{z_1} J(s, z_1 - s, z_2, \cdots, z_m)\, ds] F_m(dz_1, \cdots, dz_m).\end{aligned}$$

The last step of the calculation uses Lemma 2.1. This completes the proof.

It is from (2.2) that integrals with respect to $P^*$ will be computed. We justify this procedure by showing that certain events have uniformly small probability for all $P_t$, $t \geqq 0$.

LEMMA 2.3.

$$\lim_{t\to\infty} P_t(\{X \mid X(0+) < \gamma\}) = (h(\mu))^{-1}(\int_0^\gamma zF_1(dz) + \gamma(1 - F_1(\gamma))).$$

PROOF. We use Lemma 2.2 with $m = 1$ and $J$ the following function of *one* variable: $J(s) = 1$ if $0 \leqq s < \gamma$ and $J(s) = 0$ if $s \geqq \gamma$. Then

$$\begin{aligned}\lim_{t\to\infty} t^{-1}\int_0^t P(X(s+) < \gamma)\, ds &= \lim_{t\to\infty} t^{-1}\int_0^t E(J(X(s)))\, ds \\ &= E\lim_{t\to\infty} t^{-1}\int_0^t J(X(s))\, ds \\ &= (h(\mu))^{-1}\int_0^\infty [\int_0^{z_1} J(s)\, ds] F_1(dz_1)].\end{aligned}$$

The value of the double integral in the last line above is the value given in the statement of the lemma.

LEMMA 2.4. *Let* $\epsilon > 0$. *Then*

$$\lim_{\gamma\to\infty} \sup_{t\geqq\epsilon} P_t(\{X \mid X(0+) \geqq \gamma\}) = 0.$$

PROOF. Since $P(\{X \mid X(s+) \geqq \gamma\})$ is a bounded and measurable function of the real variable $s$, it follows that $P_t(\{X \mid X(0+) \geqq \gamma\})$ is a continuous function of $t \in [\epsilon, \infty)$. By Lemma 2.3 we may close the interval and consider these functions as continuous functions of $t \in [\epsilon, \infty]$. Again by Lemma 2.3, $P_\infty(\{X \mid X(0+) \geqq \gamma\})$ is a decreasing function of $\gamma$, while for finite values of $t$ we know this already. By Dini's theorem the conclusion of the lemma now follows.

$\{X \mid X \varepsilon K_+ \text{ and } X(0+) \leqq \gamma\}$ is a compact subset of $K_+$. If we consider, for each $\gamma > 0$, the collection $\{P_t, t > 0\}$ as a collection of continuous linear functionals on the space of continuous real valued functions on $\{X \mid X \varepsilon K_+$ and $X(0+) \leqq \gamma\}$ then there exist weak limit points. Using Lemma 2.4 and letting $\gamma \to \infty$ it follows that $\{P_t, t > 0\}$ as a set of continuous linear functionals on the bounded real valued continuous functions defined on $K_+$ has a weak limit point $P^*$ which is a probability measure. We will show that $P^*$ is uniquely determined by computing the value of the integral of certain functions.

LEMMA 2.5. *If $m \geqq 1$ and $J(\cdot)$ is a Borel measurable real valued function of $m + 1$ variables then for $F_m$ as in Lemma* 2.1,

$$(2.3) \quad \int J(X(0+), Z_0(X), \cdots, Z_{m-1}(X))\, dP^* \\ = (h(\mu))^{-1} \int \cdots \int [\int_0^{z_1} J(s, z_1 - s, \cdots, z_m)\, ds] F_m(dz_1, \cdots, dz_m).$$

PROOF. If we apply Lemma 2.2 to the difference of two continuous functions then we may obtain at once (see (1.3))

$$(2.4) \qquad \lim_{\epsilon \to 0} \lim_{t \to \infty} P_t(A_\epsilon^m) = 1.$$

Using the continuity of $P_t$ in the variable $t$, Dini's theorem, and (2.4) we obtain at once

$$(2.5) \qquad \lim_{\epsilon \to 0} \inf_{t \geqq 1} P_t(A_\epsilon^m) = 1.$$

Let $J(\cdot)$ be a bounded continuous real valued function on Euclidean $(m + 1)$-space such that $J(\cdot)$ is a nondecreasing function of each of its variables. On the closed set $A_\epsilon^m$, if $\gamma > 0$,

$$(2.6) \qquad \gamma^{-1} \int_0^\gamma J(X(s+), Z_0(X), \cdots, Z_{m-1}(X))\, ds$$

is a continuous function of $X$. If we integrate (2.6) with respect to the measure $P_t$ restricted to $A_\epsilon^m$, let $t \to \infty$, use the proof of Lemma 2.2, then let $\gamma \to 0$, $\epsilon \to 0$, we obtain (2.3).

If we interpret integration with respect to $F_m$ as integration by the countably additive measure determined by $F_m$, then by passing to the limit along sequences of continuous $J(\cdot)$ for which (2.3) holds, then taking differences, (2.3) may be established for the indicator functions of $(m + 1)$-dimensional rectangular parallelopipeds. By taking linear combinations and using a standard argument, (2.3) may be shown to hold for all Borel measurable functions $J(\cdot)$. That completes the proof of Lemma 2.5.

We complete the proof of the theorem by showing that $P^*$ is an invariant measure. Let $f$ be a bounded continuous function on $K_+$, and let $f$ have compact support. Then

$$\int f(T_w X)\, dP^* = \lim_{t \to \infty} t^{-1} \int_0^t \left( \int f(T_w X)\, dP_s \right) ds \\ = \lim_{t \to \infty} t^{-1} \int_0^t \left( \int f(T_{s+w} X)\, dP \right) ds$$

$$= \lim_{t\to\infty} t^{-1} \int_0^{t+w} \left( \int f(T_s X)\, dP \right) ds$$
$$= \lim_{t\to\infty} \int f(X)\, dP_{t+w}$$
$$= \int f(X)\, dP^*.$$

We use here the fact that $P^*$ is uniquely determined.

The proof is complete.

## REFERENCES

[1] Doob, J. L. (1948). Renewal theory from the point of view of the theory of probability. *Trans. Amer. Math. Soc.* **63** 422–438.

[2] Farrell, R. H. (1966). Limit theorems for stopped random walks III. Submitted to the *Ann. Math. Statist.*