A LOCAL LIMIT THEOREM FOR A CERTAIN CLASS OF RANDOM WALKS[1]

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1. Introduction. In [3] Karlin and McGregor considered random walks on the non-negative integers of the following type:

$$(1)\qquad \begin{aligned} P(X_{n+1} = j+1 \mid X_n = j) &= p_j = \tfrac{1}{2}[1 + \lambda/(j+\lambda)], \\ P(X_{n+1} = j-1 \mid X_n = j) &= q_j = \tfrac{1}{2}[1 - \lambda/(j+\lambda)], \\ & \qquad j = 0, 1, 2, \cdots, \end{aligned}$$

where λ is a real number greater than $-\frac{1}{2}$. Thus $p_0 = +1$, i.e. the origin is a reflecting barrier. Using the theory of orthogonal polynomials Karlin and McGregor were able to obtain integral representations for the n-step transition probabilities. In particular they showed that

$$(2)\qquad P(X_{n+m} = j \mid X_m = i) = \int_{-1}^{1} t^n Q_i(t) Q_j(t)\, d\psi(t) \Big/ \int_{-1}^{1} Q_j^2(t)\, d\psi(t),$$

where $d\psi(t) = c(1 - t^2)^{\lambda - \frac{1}{2}}$ and the Q_j's are the polynomials, orthogonal on $[-1, 1]$ with respect to the weight function $d\psi$, and c is a normalizing constant.

For random walks of the type (1), Lamperti (see [4]) showed that the limiting distribution of the random variables $X_n/n^{\frac{1}{2}}$ exists and moreover he gave an explicit formula for the limiting distribution. For example when $\lambda = \frac{1}{2}$, his result states that

$$(3)\qquad \lim_{n\to\infty} P(X_n/n^{\frac{1}{2}} \leqq t) = \int_0^t s e^{-s^2/2}\, ds = \Phi(t).$$

Lamperti proved (3) by using the method of moments. It is the purpose of this note to prove (3) in another way; by a method which will yield a stronger result. Our method is to exploit the integral representation (2) in order to obtain a local limit theorem for X_n. This in turn will enable us to conclude the following more delicate asymptotic formula:

$$(4)\qquad P(X_n/n^{\frac{1}{2}} \geqq t_n) \sim 1 - \Phi(t_n)$$

where t_n satisfies the following growth condition:

$$(5)\qquad \lim_{n\to\infty} (t_n^2/n^{\frac{1}{2}}) = 0.$$

A further consequence of (4) is a kind of law of the iterated logarithm for

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X_n, i.e. we shall show that

(6) $\limsup_{n\to\infty} [X_n/(2n \log\log n)^{\frac{1}{2}}] \leqq 1$ with probability one.

This is a strengthening of Lamperti's result (which he obtained for a much wider class of random walks, however) that

(7) $\limsup_{n\to\infty} (X_n/n^{\frac{1}{2}+\epsilon}) = 0$ with probability one, $\epsilon > 0$.

Finally it should be remarked that the main purpose of this paper is not to obtain delicate results concerning the random walk X_n, but rather to observe that the Karlin-McGregor integral representations (2) can be exploited to yield local limit theorems. The particular results in this paper are intended to illustrate this point.

2. The local limit theorem. In order to fix our ideas I shall prove the local limit theorem for the case $\lambda = \frac{1}{2}$, in which case the orthogonal polynomials which appear in (2) are the Legendre polynomials. It will become evident that the method to be presented below extends to other values of the parameter λ as well.

First, some notation: n will denote a non-negative integer and $\{j_n\}_1^\infty$ will denote a sequence of integers satisfying the following condition:

(8) $$\lim_{n\to\infty} (j_n/n) = 0.$$

We set $t_{j_n} = 2j_n/(2n)^{\frac{1}{2}}$, $s_{j_n} = (2j_n + 1)/(2n + 1)^{\frac{1}{2}}$ and

$$P_{2j_n}^{2n} = P(X_{2n} = 2j_n \mid X_0 = 0)$$

and similarly

$$P_{2j_n+1}^{2n+1} = P(X_{2n+1} = 2j_n + 1 \mid X_0 = 0).$$

For the case $\lambda = \frac{1}{2}$, we have the following local limit theorem:

THEOREM 1. *$(n/2)^{\frac{1}{2}} P_{2j_n}^{2n} = t_{j_n} [\exp(-\frac{1}{2}t_{j_n}^2)] (1 + \alpha_n(t_{j_n}))$ where $\lim_{n\to\infty} \alpha_n(t_{j_n}) = 0$ uniformly if t_{j_n} are bounded or even if $\lim_{n\to\infty} t_{j_n}^2/n^{\frac{1}{2}} = 0$.*

PROOF. The proof requires the following results concerning the Legendre polynomials (see [1], pp. 170–1):

(9) $$\int_{-1}^{1} Q_{2j_n}^2(t)\,dt = 2/(4j_n + 1);$$

(10) $$\int_{-1}^{1} t^{2n} Q_{2j_n}(t)\,dt = \pi^{\frac{1}{2}} 2^{-2n} (2n)!/\Gamma(1 + (n - j_n))\Gamma(\tfrac{3}{2} + (n + j_n)),$$

where Γ is of course the gamma function. We recall Stirling's formula for the Γ-function ([5], p. 97)

(11) $$\Gamma(x) \sim (2\pi)^{\frac{1}{2}} e^{-x} x^{x-\frac{1}{2}} (1 + O(1/x)) \quad \text{as} \quad x \to +\infty.$$

We now substitute (9) and (10) into (2) and obtain the following explicit formula for $P_{2j_n}^{2n}$:

(12) $$P_{2j_n}^{2n} = \tfrac{1}{2}(4j_n + 1)\pi^{\frac{1}{2}} 2^{-2n} (2n)!/(n - j_n)!\Gamma(\tfrac{3}{2} + (n + j_n)).$$

We now use Stirling's formula (11) to conclude that

$$P_{2j_n}^{2n} \sim 2j_n/n(n/(n-j_n))^{n-j_n}(n/(n+j_n))^{n+j_n}. \tag{13}$$

Our next step is to show that

$$(n/(n-j_n))^{n-j_n}(n/(n+j_n))^{n+j_n} \sim \exp\left(-\tfrac{1}{2}t_{j_n}^2\right). \tag{14}$$

This is easily done by taking the logarithm of the left hand side of (14). We obtain as a result that

$$\begin{aligned} -(n-j_n)\log(1-j_n/n) &- (n+j_n)\log(1+j_n/n) \\ &= -j_n^{\,2}/n + \textstyle\sum_{k=2}^{\infty}(k^{-1}-(2k-1)^{-1})j_n^{\,2k}/n^{2k-1} \\ &= -\tfrac{1}{2}t_{j_n}^2 + R_n(j_n), \end{aligned} \tag{15}$$

where the remainder term tends to zero, provided $\lim_{n\to\infty}(j_n^{\,4}/n^3) = 0$. It is easily seen that this is equivalent to the condition that $\lim_{n\to\infty}(t_{j_n}^2/n^{\frac{1}{2}}) = 0$. Applying these results to (13) we obtain

$$(n/2)^{\frac{1}{2}}P_{2j_n}^{2n} = t_{j_n}\exp\left(-\tfrac{1}{2}t_{j_n}^2\right)e^{R_n(j_n)}. \tag{16}$$

It is clear that we can write $e^{R_n(j_n)}$ as $1+\alpha_n(t_{j_n})$ and that $\lim_{n\to\infty}\alpha_n(t_{j_n}) = 0$ when $\lim_{n\to\infty}(j_n^{\,4}/n^3) = \lim_{n\to\infty}(t_{j_n}^2/n^{\frac{1}{2}}) = 0$.

In the same way it can be shown that

$$\tfrac{1}{2}(2n+1)^{\frac{1}{2}}P_{2j_n+1}^{2n+1} \sim s_{j_n}\exp\left(-\tfrac{1}{2}s_{j_n}^2\right). \tag{16'}$$

Let us recall the following simple inequality:

If $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ are two sequences of non-negative numbers, then

$$\min_{1\leqq i\leqq n}(a_i/b_i) \leqq (a_1+\cdots+a_n)/(b_1+\cdots+b_n) \leqq \max_{1\leqq i\leqq n}(a_i/b_i). \tag{17}$$

Using this result and setting $\phi(s) = se^{-S^2/2}$, we conclude that

$$\textstyle\sum_{u_n\leqq t_{j_n}\leqq v_n}(n/2)^{\frac{1}{2}}P_{2j_n}^{2n} \sim \sum_{u_n\leqq t_{j_n}\leqq v_n}\phi(t_{j_n}), \tag{18}$$

or equivalently,

$$P(u_n \leqq X_{2n}/(2n)^{\frac{1}{2}} \leqq v_n) = \textstyle\sum_{u_n\leqq t_{j_n}\leqq v_n}P_{2j_n}^{2n} \sim (2/n)^{\frac{1}{2}}\sum_{u_n\leqq t_{j_n}\leqq v_n}\phi(t_{j_n}), \tag{18'}$$

provided of course that $\lim_{n\to\infty}(u_n^{\,2}/n^{\frac{1}{2}}) = \lim_{n\to\infty}(v_n^{\,2}/n^{\frac{1}{2}}) = 0$. It is easy to verify (e.g. see [2], pp. 171–2) that the right hand side is a Riemann sum approximating the Riemann integral $\int_{u_n}^{v_n}\phi(t)dt$. We have thus obtained the following stronger form of Lamperti's limit theorem for $X_{2n}/(2n)^{\frac{1}{2}}$.

THEOREM 2. *Let u_n and v_n be sequences of non-negative numbers satisfying the condition $\lim_{n\to\infty}(u_n^{\,2}/n^{\frac{1}{2}}) = \lim_{n\to\infty}(v_n^{\,2}/n^{\frac{1}{2}}) = 0$, $u_n \leqq v_n$. Then*

$$P(u_n \leqq X_{2n}/(2n)^{\frac{1}{2}} \leqq v_n) \sim \int_{u_n}^{v_n}\phi(t)dt.$$

It can be shown, in a similar way, that

$$\lim_{n\to\infty} P(u_n \leqq X_{2n+1}/(2n+1)^{\frac{1}{2}} \leqq v_n) \sim \int_{u_n}^{v_n} \phi(t)dt,$$

where u_n and v_n satisfy the same conditions as before.

3. A theorem on large deviations. We are now in a position to prove the following important result:

THEOREM 3. *If t_n is a sequence of numbers tending to $+\infty$ and such that $\lim_{n\to\infty} (t_n^2/n^{\frac{1}{2}}) = 0$, then*

$$P(X_n/n^{\frac{1}{2}} \geqq t_n) \sim 1 - \Phi(t_n). \tag{19}$$

PROOF. A similar theorem for the Bernoulli trials process is well known (see [2], pp. 178–179) and the proof given here is in the same spirit. First we form the ratio

$$\rho_n(k) = P_{2k+2}^{2n}/P_{2k}^{2n}. \tag{20}$$

Using formula (12) we can show, after some elementary algebraic manipulations, that

$$\rho_n(k) = 2[1 + 4/(4k+1)](n-k)/(2(n+k)+1). \tag{21}$$

It is also easily seen that (i) $\rho_n(k)$ is monotone decreasing as k increases for fixed n and (ii) $\rho_n(k) < 1$ if $k \geqq n^{\frac{1}{2}}$, this last inequality is easily verified by a direct computation. Thus we conclude that for $j \geqq n^{\frac{1}{2}}$,

$$P(X_{2n} \geqq 2j) = \sum_{m=0}^{n-j} P_{2j+2m}^{2n} \leqq \sum_{m=0}^{n-j} P_{2j}^{2n} \rho_n{}^m(j) \leqq P_{2j}^{2n}\{1 - \rho_n(j)\}^{-1}. \tag{22}$$

More algebraic computations of a simple kind show that $\{1 - \rho_n(j)\}^{-1} = (4j+1)(n+j+\frac{1}{2})/(8j^2 - 3j - 5n)$. If we now assume that $j_n \to +\infty$ and also that $\lim_{n\to\infty} (j_n^4/n^3) = 0$, then it follows from Theorem 2 that

$$(2n)^{\frac{1}{2}} P_{2j_n}^{2n} (2n)^{-\frac{1}{2}}\{1 - \rho_n(j_n)\}^{-1} \sim 2\phi(t_{j_n})2t_{j_n}/(4t_{j_n}^2 - 5) \sim \phi(t_{j_n})/t_{j_n}, \tag{23}$$

because we are assuming that $\lim_{n\to\infty} t_{j_n} = +\infty$. Moreover, as is easily verified, $1 - \Phi(t) = \phi(t)/t$. Hence we have shown that

$$P(X_{2n} \geqq 2j_n) = P(X_{2n}/(2n)^{\frac{1}{2}} \geqq t_{j_n}) = O(1 - \Phi(t_{j_n})), \tag{24}$$

where $\lim_{n\to\infty} (t_{j_n}^2/n^{\frac{1}{2}}) = 0$ and $\lim_{n\to\infty} t_{j_n} = +\infty$.

The proof of Theorem 3 is now completed as follows: if $\lim_{n\to\infty} (t_n^2/n^{\frac{1}{2}}) = 0$ then $\lim_{n\to\infty} [(t_n + \log t_n)^2/n^{\frac{1}{2}}] = 0$ also. Set $u_n = t_n + \log t_n$. Then by Theorem 2 we have

$$P(t_n \leqq X_{2n}/(2n)^{\frac{1}{2}} \leqq u_n) \sim 1 - \Phi(t_n) - (1 - \Phi(u_n)). \tag{25}$$

Moreover, by direct computation, we have $\lim_{n\to\infty} (1 - \Phi(u_n))/(1 - \Phi(t_n)) = 0$. Also by (24) we have that $P(X_{2n}/(2n)^{\frac{1}{2}} \geqq u_n) = O(1 - \Phi(u_n))$. Putting these results together we conclude that

$$P(X_{2n}/(2n)^{\frac{1}{2}} \geqq t_n) \sim 1 - \Phi(t_n), \qquad \lim_{n\to\infty} (t_n^2/n^{\frac{1}{2}}) = 0. \tag{26}$$

In a similar way we can show that

$$(26') \quad P(X_{2n+1}/(2n+1)^{\frac{1}{2}} \geqq t_n) \sim 1 - \Phi(t_n), \qquad \lim_{n\to\infty} (t_n^{\,2}/n^{\frac{1}{2}}) = 0.$$

We now conclude this paper with a few remarks concerning the so-called "law of the iterated logarithm."

In order to prove that $\limsup_{n\to\infty} [X_n/(2n \log \log n)^{\frac{1}{2}}] \leqq 1$ with probability one it is sufficient to show that for any $\lambda > 1$ the event $\{X_n > \lambda.\,(2n \log \log n)^{\frac{1}{2}}\}$ occurs infinitely often with probability zero. This can be done in almost the same way that Feller proves the law of the iterated logarithm for a Bernoulli trials process (see [2], pp. 192–195). In this proof the asymptotic result (26) plays a crucial role. We omit the details.

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