

NOTES

A SAMPLE FUNCTION PROPERTY OF MARTINGALES¹

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We show that if x_n is an L_1 bounded martingale then $\sum_n (x_n - x_{n-1})^2$ converges almost everywhere. We refer to Doob's book [1], Chapter VII, for background material.

THEOREM. *Let $\dots x_{-1}, x_0, x_1, \dots$ be a martingale, defined on some probability space, such that $\sup_n \int |x_n| = K < \infty$. Then $\sum_{n=-\infty}^{\infty} (x_n - x_{n-1})^2 < \infty$ almost everywhere.*

PROOF. Let $E_n = \bigcap_{k \leq n} [w; |x_k(w)| < M]$, $E = \bigcap_n E_n$, where M is a positive real number. Define \hat{x}_n by

$$\begin{aligned} \hat{x}_n(w) &= x_n(w) && \text{if } |x_n(w)| < M, \\ &= 0 && \text{if } |x_n(w)| \geq M. \end{aligned}$$

Then

$$\int_{E_{n-1}} x_n x_{n-1} = \int_{E_{n-1}} x_n^2 = \int_{E_{n-1}} \hat{x}_n^2,$$

implying that

$$\int_{E_{n-1}} (\hat{x}_n - \hat{x}_{n-1})^2 = \int_{E_{n-1}} (\hat{x}_n^2 - \hat{x}_{n-1}^2) + 2 \int_{E_{n-1}} (x_n x_{n-1} - \hat{x}_n \hat{x}_{n-1}).$$

Let N be a positive integer. Then

$$\begin{aligned} \sum_{|n| \leq N} \int_{E_{n-1}} (\hat{x}_n^2 - \hat{x}_{n-1}^2) &= \int_{E_{N-1}} \hat{x}_N^2 + \sum_{-N \leq n < N} \int_{E_{n-1} - E_n} \hat{x}_n^2 \\ &\quad - \int_{E_{-N-1}} \hat{x}_{-N-1}^2 \\ &\leq M^2 \end{aligned}$$

Using $x_{n-1} = \hat{x}_{n-1}$ on E_{n-1} , $|x_n - \hat{x}_n| \leq |x_n|$, and the fact that $\dots |x_{-1}|, |x_0|, \dots$ is a submartingale, we have that

$$\begin{aligned} \int_{E_{n-1}} (x_n x_{n-1} - \hat{x}_n \hat{x}_{n-1}) &\leq M \int_{E_{n-1}} |x_n - \hat{x}_n| \\ &= M \int_{E_{n-1} - E_n} |x_n - \hat{x}_n| \\ &\leq M \int_{E_{n-1} - E_n} |x_n| \\ &\leq M \int_{E_{n-1} - E_n} |x_N|, \end{aligned}$$

for all $n \leq N$, implying that

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$$\sum_{|n| \leq N} \int_{E_{n-1}} (x_n x_{n-1} - \hat{x}_n \hat{x}_{n-1}) \cong M \int |x_N| \cong MK.$$

Therefore,

$$\begin{aligned} \int_E \sum_n (x_n - x_{n-1})^2 &= \sum_n \int_E (\hat{x}_n - \hat{x}_{n-1})^2 \\ &\cong \sum_n \int_{E_{n-1}} (\hat{x}_n - \hat{x}_{n-1})^2 \\ &\cong M^2 + 2MK < \infty, \end{aligned}$$

and $\sum_n (x_n - x_{n-1})^2$ is finite almost everywhere on E . But for large M the measure of E is near 1 since, by the martingale convergence theorem, $\sup_n |x_n| < \infty$ almost everywhere. This completes the proof.

REFERENCE

- [1] Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.