# TEST PROCEDURES FOR POSSIBLE CHANGES IN PARAMETERS OF STATISTICAL DISTRIBUTIONS OCCURRING AT UNKNOWN TIME POINTS

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**0. Introduction and summary.** The present study is concerned with the properties of a test statistic proposed by H. Chernoff and S. Zacks [1] to detect shifts in a parameter of a distribution function, occurring at unknown time points between consecutively taken observations. The testing problem we study is confined to a fixed sample size situation, and can be described as follows: Given observations on independent random variables $X_1, \cdots, X_n$, (taken at consecutive time points) which are distributed according to $F(X; \theta_i)$; $\theta_i \varepsilon \Omega$ for all $i = 1, \cdots, n$, one has to test the simple hypothesis: $H_0 : \theta_1 = \cdots = \theta_n = \theta_0$ ($\theta_0$ is known) against the composite alternative:

$$H_1 : \theta_1 = \cdots = \theta_m = \theta_0$$

$$\theta_{m+1} = \cdots = \theta_n = \theta_0 + \delta; \qquad \delta > 0,$$

where both the point of change, $m$, and the size of the change, $\delta$, are unknown $(m = 1, \cdots, n - 1), 0 < \delta < \infty$.

A Bayesian approach led Chernoff and Zacks in [1] to propose the test statistic $T_n = \sum_{i=1}^{n-1} iX_{i+1}$, for the case of normally distributed random variables. A generalization for random variables, whose distributions belong to the one parameter exponential family, i.e., their density can be represented as $f(x; \theta) = h(x) \exp [\psi_1(\theta)U(x) + \psi_2(\theta)]$, $\theta \varepsilon \Omega$ where $\psi_1(\theta)$ is monotone, yields the test statistic $T_n = \sum_{i=1}^{n-1} iU(x_{i+1})$. In the present paper we study the operating characteristics of the test statistic $T_n$. General conditions are given for the convergence of the distribution of $T_n$, as the sample size grows, to a normal distribution. The rate of convergence is also studied. It was found that the closeness of the distribution function of $T_n$ to the corresponding normal distribution is not satisfactory for the purposes of determining test criteria and values of power functions, in cases of small samples from non-normal distributions. The normal approximation can be improved by considering the first four terms in Edgeworth's asymptotic expansion of the distribution function of $T_n$ (see H. Cramér [2] p. 227). Such an approximation involves the normal distribution, its derivatives and the semi-invariants of $T_n$. The goodness of the approximations based on such an expansion, and that of the simple normal approximation, for small sample situations, were studied for cases where the observed random variables are binomially or exponentially distributed. In order to compare the exact distribution functions of $T_n$ to the approximations, the exact forms of the distributions of $T_n$ in the bi-

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nomial and exponential cases were derived. As seen in Section 4, these distribution functions are quite involved, especially under the alternative hypothesis. Tables of coefficients are given for assisting the determination of these distributions, under the null hypothesis assumption, in situations of samples whose size is $2 \leqq n \leqq 10$. For samples of size $n \geqq 10$ one can use the normal approximation to the test criterion and obtain good results. The power functions of the test statistic $T_n$, for the binomial and exponential cases, are given in Section 5. The comparison with the values of the power function obtained by the normal approximation is also given. As was shown by Chernoff and Zacks in [1], when $X$ is binomially distributed the power function values of $T_n$ are higher than those of a test statistic proposed by E. S. Page [5], for most of the $m$ values (points of shift) and $\delta$ values (size of shift). A comparative study in which the effectivenes of test procedures based on $T_n$ relative to those based on Page's and other procedures will be given elsewhere for the exponential case, and other distributions of practical interest.

**1. Bayes procedures for the exponential family.** Let $X_1, \cdots, X_n$ be a sequence of independent random variables, whose density functions belong to the one parameter exponential family,

$$f(x;\theta) = h(x) \exp [\psi_1(\theta)U(x) + \psi_2(\theta)], \qquad \theta \varepsilon \Omega \tag{1.1}$$

where $\psi_1(\theta)$ is a monotonically increasing function of $\theta$.

We further assume that $\psi_1(\theta)$ and $\psi_2(\theta)$ have finite derivatives on $\Omega$. The problem is to test the simple hypothesis that all random variables are identically distributed, with $\theta = \theta_0$ (known) against the composite alternative that at some point $m$ (unknown) a shift has occurred at the value of the parameter. That is

$$\begin{aligned} &H_0 : \theta_1 = \cdots = \theta_n = \theta_0 && (\theta_0 \text{ is known}) \\ &H_1 : \theta_1 = \cdots = \theta_m = \theta_0 ; \quad \theta_{m+1} = \cdots = \theta_n = \theta_0 + \delta, && 0 < \delta \end{aligned} \tag{1.2}$$

$m$ is unknown ($m = 1, 2, \cdots, n - 1$); $\delta$ is unknown, $\theta_0 + \delta \varepsilon \Omega$.

Following Chernoff and Zacks [1], we derive a test statistic for the given problem, according to the following Bayesian approach. Consider the point of change $m$ as a realization of a random variable $M$, having the following prior density.

$$\begin{aligned} \Pi_M(m) &= (n-1)^{-1}, && \text{if } m = 1, 2, \cdots, n-1 \\ &= 0, && \text{otherwise.} \end{aligned} \tag{1.3}$$

The marginal likelihood function of the sample, under the alternative $H_1$ is

$$\begin{aligned} f_1(x_1, \cdots, x_n ; \theta_0, \delta) = \textstyle\prod_{i=1}^n h(X_i)\cdot(n-1)^{-1} \sum_{m=1}^{n-1} \exp [\sum_{i=1}^m \eta(X_i, \theta_0)] \\ \cdot \exp [\textstyle\sum_{i=m+1}^n \eta(X_i ; \theta_0 + \delta)] \end{aligned} \tag{1.4}$$

where $\eta(X_i ; \theta) = \psi_1(\theta)U(X_i) + \psi_2(\theta)$, $\theta \varepsilon \Omega$. Since the derivatives of $\psi_1(\theta)$ and $\psi_2(\theta)$ are finite on $\Omega$ one can write, in a close neighborhood of $\theta_0$,

$$\eta(X_i ; \theta_0 + \delta) = \eta(X_i ; \theta_0) + \delta\eta'(X_i ; \theta_0) + o(\delta), \tag{1.5}$$

for all $i = 1, \cdots, n$. The marginal likelihood function under $H$, is then,

$$(1.6)\quad f_1(X_1, \cdots, X_n; \theta_0, \delta) = \prod_{i=1}^{n} h(X_i) \exp [\eta(X_i; \theta_0)] \cdot (n-1)^{-1} \sum_{m=1}^{n-1} [1 + \delta \sum_{i=m+1}^{n} \eta'(x_i; \theta_0)] + o(\delta), \quad \text{as } \delta \to 0.$$

It follows that the likelihood ratio can be expressed, as $\delta \to 0$, by

$$(1.7)\quad L(X_1, \cdots, X_n; \theta_0, \delta) = 1 + [\delta/(n-1)]\, \psi_1'(\theta_0) \sum_{i=1}^{n-1} iU(X_{i+1}) + \delta \tfrac{1}{2} n \psi_2'(\theta_0) + o(\delta)$$

since the likelihood function under $H_0$ is given by the left hand factor of (1.6). A Bayes procedure for testing $H_0$ against $H_1$ is to reject $H_0$ whenever the likelihood ratio is larger than a suitably chosen test criterion (which depends on the loss function and the prior probability that $H_0$ is true). Thus, since $\psi_1(\theta)$ is monotonically increasing, a Bayes test statistic for the above problem is $T_n = \sum_{i=1}^{n-1} iU(X_{i+1})$. $H_0$ is rejected whenever $T_n$ is larger than a suitable test criterion. We notice that:

(i) $T_n$ is a linear function of the sufficient statistics of the underlying distributions, namely $U(X_i)$, $i = 2, \cdots, n$.

(ii) $T_n$ is independent of $X_1$, since the distribution of $X_1$ is known.

(iii) The last observations attain a higher weight than the first ones. The weights assigned to the observations depend upon the prior distribution chosen for the point of change, $M$. The statistic $T_n = \sum_{i=1}^{n-1} iU(X_{i+1})$ is derived under the assumption of uniform prior probabilities for the possible values of $M$. If we assign to the values of $M$ arbitrary prior probabilities, $\Pi_M(m)$, $m = 1, \cdots, n-1$; then a similar analysis will yield the test statistic $T_n^{(\pi)} = \sum_{i=1}^{n-1} \omega_i(\pi) \cdot U(X_{i+1})$ where $\omega_{n-1}(\pi) \propto 1$ and $\omega_i(\pi) \propto \sum_{m=1}^{i} \Pi_M(m) = P^{(\pi)}[M \leqq i]$. That is, the weight assigned to $U(X_{i+1})$ is proportional to the prior probability that the shift in $\theta$ occurred before the $(i+1)$st observation.

(iv) All the procedures based on the test statistics $T_n^{(\pi)}$ are admissible. They all maximize the derivative of the average power when $\delta = 0$.

(v) The test statistics $T_n^{(\pi)}$ can be obtained also by assigning the amount of change, $\delta$, a prior exponential distribution over $[0, \infty]$ with expectation $1/t$. The likelihood ratio (1.6) becomes then, as $t \to \infty$

$$(1.8)\quad \begin{aligned} &L(X_1, \cdots, X_n; \theta_0, t) \\ &= \sum_{m=1}^{n-1} \Pi_M(m)[1 - (1/t) \sum_{i=m+1}^{n} \eta'(X_i; \theta_0)]^{-1} + o(1/t) \\ &= 1 + (\psi_1'(\theta_0)/t) \sum_{m=1}^{n-1} \Pi_M(m) \sum_{i=m+1}^{n} U(X_i) + (\psi_2'(\theta_0)/t) + o(1/t) \\ &= 1 + (\psi_1'(\theta_0)/t) T_n^{(\pi)} + (\psi_2'(\theta_0)/t) + o(1/t). \end{aligned}$$

Since the expectation of the exponential prior distribution, $1/t$, approaches zero when $t \to \infty$ we obtain by the present Bayesian approach the same test statistics as by the former one. This shows that the test statistic $T_n$ is robust with respect to the prior assumptions according to which it is derived.

**2. The asymptotic distribution of $T_n$.** The exact distribution function of $T_n = \sum_{i=1}^{n-1} iU(X_{i+1})$ is very complicated in many cases where $U(X)$ is not normally distributed. The asymptotic distribution of $T_n$ is, however, under very general conditions, a normal distribution. The following theorem specifies these conditions.

THEOREM. *Let $X_1, X_2, \cdots$ be a sequence of independent random variables, having the corresponding* cdf's $F(X; \theta_i)$, $\theta_i \varepsilon \Omega$ *for all* $i = 1, 2, \cdots$. *If*

$$E_{\theta_i}\{|U(X_i)|^3\} \leqq M < \infty, \qquad \textit{for all } i = 1, 2, \cdots \tag{2.1}$$

*and*

$$\mathrm{Var}_{\theta_i}\{U(X_i)\} \geqq \epsilon > 0, \qquad \textit{for all } i = 1, 2, \cdots \tag{2.2}$$

*then,*

$$\lim_{n\to\infty} P[(T_n - E(T_n))/(\mathrm{Var}\ T_n)^{\frac{1}{2}} \leqq z] = \Phi(z), \quad -\infty < z < \infty \tag{2.3}$$

*where $\Phi(\cdot)$ is the standard normal integral.*

PROOF. Let $\beta_i(\theta_i) = E_{\theta_i}\{|U(X_i) - E_{\theta_i}U(X_i)|^3\}$. Condition (2.1) implies that the sequence $\{\beta_i(\theta_i)\}$ is bounded. Condition (2.2) implies that $\lim_{n\to\infty} \sum_{i=1}^n i^2 \mathrm{Var}_{\theta_i}\{U(X_i)\} = \infty$. Finally, since

$$\lim_{n\to\infty} (\textstyle\sum_{i=1}^n i^3\beta_i(\theta_i))^{\frac{1}{3}}/(\sum_{i=1}^n i^2 \mathrm{Var}_{\theta_i}\{U(X_i)\})^{\frac{1}{2}} \leqq \lim_{n\to\infty} O(n^{-1/6}) = 0, \tag{2.4}$$

all the Liaponnoff's conditions are satisfied, and by the central limit theorem (see M. Fisz [4], p. 202) (2.3) holds.

**3. The rate of convergence of the distribution of $T_n$ to the normal distribution.** As proven by H. Cramér [3] Ch. 7, Theorems 24, 26 if $Y_1, Y_2, \cdots$ is a sequence of independent random variables such that, all the expectation of $Y_1, Y_2, \cdots$ are zero, and their third absolute moments are finite then,

$$|F_n(z) - \Phi(z)| < C \log n \textstyle\sum_{i=1}^n E|Y_i|^3/(\sum_{i=1}^n \sigma_i^2)^{\frac{3}{2}}, \quad -\infty < z < \infty \tag{3.1}$$

where $C$ is a constant independent of $z$ and $n$, where $\sigma_i^2$ denotes the variance of $Y_i$ $(i = 1, 2, \cdots)$; and where $F_n(z)$ is the cdf of the standardized random variable $Z_n = \sum_{i=1}^n (Y_i - EY_i)/(\sum_{i=1}^n \sigma_i^2)^{\frac{1}{2}}$. For the present problem, define $Y_i = i[U(X_i) - E_{\theta_i}U(X_i)]$ $(i = 1, 2, \cdots)$. Under Conditions (2.1) and (2.2) one obtains,

$$\textstyle\sum_{i=1}^n i^3 E_{\theta_i}|U(X_i) - E_{\theta_i}U(X_i)|^3/(\sum_{i=1}^n i^2 \mathrm{Var}_{\theta_i}\{U(X_i)\})^{\frac{3}{2}} = O(n^{-\frac{1}{2}}) \tag{3.2}$$

as $n \to \infty$. Hence, from (3.1) and (3.2), the rate of convergence of $P[(T_n - ET_n)/(\mathrm{Var}\ T_n)^{\frac{1}{2}} \leqq z]$ to $\Phi(z)$ is like that of $n^{-\frac{1}{2}} \log n$ to zero, i.e.,

$$|F_n(z) - \Phi(z)| = O((\ln n)/n^{\frac{1}{2}}), \qquad \text{as } n \to \infty \cdot \tag{3.3}$$

This could be, however, a slow rate of convergence. A better approximation to the distribution of $Z_n$, for moderate sample sizes, can be attempted by the expanding of the distribution function $F_n(z)$ in the form,

$$(3.4)\quad F_n(z) = \Phi(z) + \sum_{v=1}^{k-3} n^{-v/2}\left[\sum_{j=1}^{v} (-1)^{v+2j} C_{jvn}\Phi^{(v+2j)}(z)\right] + R_{kn}(z)$$

where $\Phi^{(v)}(z)$ is the $v$th derivative of $\Phi(z)$; $C_{jvn}$ are polynomials of the semi-invariants of $T_n$; $R_{kn}(z)$ is the remainder term (see H. Cramér [2] p. 228). For most practical purposes it is sufficient to consider expansion (3.4) up to $k = 3$. Denote by $\mu_{k,n}^*$ the $k$th central moment of $T_n$, and let $\gamma_{1,n} = \mu_{3,n}^*/(\mu_{2,n}^*)^{\frac{3}{2}}$, $\gamma_{2,n} = \mu_{4,n}^*/(\mu_{2,n}^*)^2 - 3$ (the coefficients of asymmetry and kurtosis). Expansion (3.4) is then,

$$(3.5)\quad F_n(z) = \Phi(z) - (\gamma_{1,n}/3!)\Phi^{(3)}(z) + (\gamma_{2,n}/4!)\Phi^{(4)}(z) + (10\gamma_{1,n}^2/6!)\Phi^{(6)}(z) + R_{5,n}(z).$$

The order of magnitude of the remainder term is $R_{5,n}(z) = O(n^{-\frac{3}{2}})$ as $n \to \infty$ (see H. Cramér [3] Theorem 26).

The coefficients $\gamma_{1,n}$ and $\gamma_{2,n}$ depend on $\theta_0$, the point of shift $m$, and the size of the shift $\delta$. Under conditions of the hypothesis, $H_0$, the coefficients $\gamma_{1,n}$ and $\gamma_{2,n}$ depend only on $\theta_0$. We denote these coefficients, under the hypothesis $H_0$, by $\gamma_{1,n}^{(0)}$ and $\gamma_{2,n}^{(0)}$ and determine them according to the formulae:

$$(3.6)\quad \gamma_{1,n}^{(0)} = 3[3(n-1)n/2(2n-1)^3]^{\frac{1}{2}}\{\mu_3^*(U(X))/[\mu_2^*(U(X))]^{\frac{3}{2}}\}$$

and

$$(3.7)\quad \gamma_{2,n}^{(0)} = \tfrac{6}{5}[(3n(n-1)-1)/(n-1)n(2n-1)] \cdot [\mu_4^*(U(X))/[\mu_2^*(U(X))]^2] - 3$$

where $\mu_k^*(U(X))$ designates the $k$th central moment of $U(X)$. Numerical analysis shows that, for almost all values of $n$, one can approximate $\gamma_{1,n}$ and $\gamma_{2,n}$ by

$$(3.8)\quad \gamma_{1,n}^{(0)} \cong 1.3n^{-\frac{1}{2}}\gamma_1(U(X))$$

and

$$(3.9)\quad \gamma_{2,n}^{(0)} \cong 1.8n^{-1}\gamma_2(U(X))$$

where $\gamma_i(U(X))$ $(i = 1, 2)$ are the coefficients of asymmetry and kurtosis of $U(X)$. Thus, for further numerical analysis we shall use the following approximation to $F_n^{(0)}(z)$ (the distribution of $T_n$ under $H_0$)

$$(3.10)\quad F_n^{(0)}(z) \cong \Phi(z) - (1.3/n^{\frac{1}{2}})\gamma_1(U(X))[\Phi^{(3)}(z)/3!] + [(1.8/n)\gamma_2(U(X))[\Phi^{(4)}(z)/4!] + (17/n)\gamma_1^2(U(X))[\Phi^{(6)}(z)/6!]].$$

Denote by $F_n^*(z)$ the right hand side of (3.10).

In the rest of this section we study numerically the goodness of the normal approximation $\Phi(z)$ to $F_n^*(z)$, in small sample situations, under binomial and exponential distributions. It is assumed that no shift in the parameter $\theta_0$ occurred.

(i) *The binomial case.* We study the binomial case where the observed random variable is assigned the values $+1$ or $-1$ with probabilities $p$ and $1 - p$ respectively. This case is under consideration, for example, in a nonparametric sign test.

The density function of $X$ can be written as,

$$f(x; p) = p^{(1+x)/2}(1 - p)^{(1-x)/2}, \qquad x = \pm 1$$
$$= 0, \qquad \text{otherwise.}$$

Or, in the exponential form:

$$f(x; p) = \exp\{\tfrac{1}{2}[\ln p - \ln(1 - p)]x + \tfrac{1}{2}[\ln p + \ln(1 - p)]\}, \qquad x = \pm 1$$
$$= 0, \qquad \text{otherwise.}$$

Therefore, the test statistic is $T_n = \sum_{i=1}^{n-1} iX_{i+1}$. The $k$th central moment of $X$ is given by the formula $\mu_k^* = 2^k p(1 - p)[(1 - p)^{k-1} - (-p)^{k-1}]$, $k = 0, 1, 2, \cdots$. The coefficients of asymmetry and kurtosis are, in the present binomial case, $\gamma_1 = (1 - 2p)/[p(1 - p)]^{\frac{1}{2}}$ and $\gamma_2 = ((1 - p)^3 + p^3)/p(1 - p) - 3$. The distribution of $X$ is closest to a normal one when $p = .5$. As $p$ grows from .5 to 1.0 the coefficient $\gamma_1$ is decreasing monotonically from zero to $-\infty$. The coefficient of kurtosis $\gamma_2$ increases monotonically from $-2$ to $\infty$. We conclude that the normal approximation $\Phi(z)$ to $F_n^*(z)$ becomes less and less effective as $p$ approaches 1 (or zero). Table 3.1 represents some numerical comparisons of $\Phi(z)$ to $F_n^*(z)$. We expect $F_n^*(z)$ to be very close to the exact distribution of $Z_n$.

The following conclusions can be drawn from the present table:

(i) When $p = 0.5$ the normal approximation to the distribution of $T_n$ under $H_0$ is good even in a small sample of size $n = 5$; and is certainly an excellent approximation if $n \geqq 10$.

(ii) When $p$ is as large as 0.9 the normal approximation requires a large sample. Even in the case of $n = 20$ the normal approximation might not be good enough.

TABLE 3.1

*Values of $F_n^*(z) - \Phi(z)$ in the binomial case; $n = 5, 10, 20$; $p = 0.5, 0.9$*

| $z$ | $1 - \Phi(z)$ | $n = 5$ | | $n = 10$ | | $n = 20$ | |
|---|---|---|---|---|---|---|---|
| | | $p = 0.5$ | $p = 0.9$ | $p = 0.5$ | $p = 0.9$ | $p = 0.5$ | $p = 0.9$ |
| 2.326 | 0.010 | 0.004 | 0.043 | 0.002 | 0.023 | 0.002 | 0.018 |
| 1.960 | 0.025 | 0.002 | 0.074 | 0.000 | 0.043 | 0.000 | 0.028 |
| 1.645 | 0.050 | −0.002 | 0.079 | 0.000 | 0.052 | 0.000 | 0.028 |
| 1.282 | 0.100 | −0.010 | 0.055 | −0.004 | 0.031 | −0.002 | 0.018 |
| 0.674 | 0.250 | −0.016 | −0.083 | −0.008 | −0.048 | −0.004 | −0.025 |
| 0.000 | 0.500 | 0.000 | −0.104 | 0.000 | −0.072 | 0.000 | −0.051 |
| −0.674 | 0.750 | 0.016 | −0.007 | 0.008 | −0.016 | 0.004 | −0.017 |
| −1.282 | 0.900 | 0.010 | 0.003 | 0.004 | 0.011 | 0.002 | 0.008 |

TABLE 3.2

*Values of $F_n^*(z) - \Phi(z)$ in the exponential case; $n = 5, 10, 20$*

| $z$ | $1 - \Phi(z)$ | $n = 5$ | $n = 10$ | $n = 20$ |
|---|---|---|---|---|
| 2.326 | 0.010 | −0.024 | −0.018 | −0.014 |
| 1.960 | 0.025 | −0.018 | −0.014 | −0.012 |
| 1.645 | 0.050 | −0.022 | −0.016 | −0.012 |
| 1.282 | 0.100 | 0.008 | −0.004 | −0.004 |
| 0.674 | 0.250 | 0.038 | 0.028 | 0.020 |
| 0.000 | 0.500 | 0.078 | 0.054 | 0.038 |
| −0.674 | 0.750 | 0.030 | 0.020 | 0.012 |
| −1.282 | 0.900 | −0.052 | −0.028 | −0.016 |

(ii) *The exponential case.* If the distribution of $X$ is exponential with intensity $\lambda$, $0 < \lambda < \infty$. The test statistic is also $T_n = \sum_{i=1}^{n-1} iX_{i+1}$. The coefficients of asymmetry and kurtosis under $H_0$ are independent of $\lambda$ and are equal to $\gamma_1 = 2$ and $\gamma_2 = 6$. The goodness of the normal approximation depends only on the sample size $n$. In the following table the deviations of $F_n^*(z)$ from $\Phi(z)$ are given for $n = 5$, 10 and 20. $F_n^*(z)$ is given by the righthand side of formula (3.11).

As seen in Table 3.2, the normal approximations is very ineffective for small samples. Even samples of size $n = 20$ are not large enough for this purpose and it is suggested that the approximating formula in small sample situations will be according to (3.10), when substituting $\gamma_1 = 2$ and $\gamma_2 = 6$,

$$(3.11)\quad F_n^{(0)}(z) \cong \Phi(z) - (0.433/n)\Phi^{(3)}(z) + (0.450/n)\Phi^{(4)}(z) + (0.094/n)\Phi^{(6)}(z).$$

A similar approximation for the case of a shift occurring after the $m$th observation will be given in Section 5.

**4. The exact distributions of the test statistic $T_n$ in the binomial and exponential cases.** In the present section we present the method of derivation and the formulae of the exact distributions of $T_n$, under $H_0$ and under $H_1$. These formulae are basic for the computations in Section 5.

(i) *The binomial case.* As in the previous example we consider here the binomial case where $X$ has the density

$$f(x; p) = p^{(1+x)/2}(1 - p)^{(1-x)/2}, \qquad x = \pm 1$$
$$= 0, \qquad \text{otherwise.}$$

Let $X_1, \cdots, X_n$ be a sequence of independent random variables having the above binomial distribution with parameters $p_1, \cdots, p_n$ respectively. Consider the hypothesis $H_0 : p_1 = \cdots = p_n = \frac{1}{2}$; and the alternative $H_1 : p_1 = \cdots = p_m = \frac{1}{2}$, $p_{m+1} = \cdots = p_n = \frac{1}{2} + \delta$, where $m$ (unknown) represents the point of shift, and $\delta$ represents the amount of shift, $-\frac{1}{2} < \delta < \frac{1}{2}$. Let $G_{T_n}^{(m,\theta)}(u)$ designate the generating function of $T_n$, under $H_1$, where $\theta = \frac{1}{2} + \delta$. It is easy to prove that,

$$G_{T_n}^{(m,\theta)}(u) = \prod_{i=0}^{m-1} (\tfrac{1}{2}u^i + \tfrac{1}{2}u^{-i}) \prod_{i=m}^{n-1} (\theta u^i + (1-\theta)u^{-i})$$
$$(4.1) \qquad = [(1-\theta)^{n-m}/(2^m u^{n(n-1)/2})]$$
$$\cdot \prod_{i=0}^{m-1} (1 + u^{2i}) \prod_{i=m}^{n-1} [1 + (\theta/(1-\theta))u^{2i}]$$

for all $u \neq 0$, $0 < \theta < 1$, $m = 1, \cdots, n-1$. Let $G_{T_n}^{(0)}(u)$ denote the generating function of $T_n$, under $H_0$. We have,

$$(4.2) \quad G_{T_n}^{(0)}(u) = 2^{-(n-1)} \prod_{i=1}^{n-1} (u^i + u^{-i}) = [1/(2^{n-1}u^{n(n-1)/2})] \prod_{i=1}^{n-1} (1 + u^{2i})$$

for $u \neq 0$.

The probability density function of $T_n$ under $H_1$ is obtained from the coefficients of the polynomial expansion of $G_{T_n}^{(\theta,m)}(u)$, at $u = 1$. Define

$$(4.3) \qquad h_{n-1}^{(m,a)}(s) = \prod_{i=1}^{m-1} (1 + s^i) \prod_{j=m}^{n-1} (1 + as^j)$$
$$= \sum_{r=0}^{n(n-1)/2} b_r(a, m)s^r$$

where $a = \theta/(1-\theta)$, $s = u^2$. The polynomials $h_{n-1}^{(m,a)}(s)$ can be easily determined according to the following recursive formula:

$$(4.4) \qquad h_{k+1}^{(m,a)}(s) = h_k^{(m,a)}(s)[1 + a_{k+1,m}s^{k+1}], \qquad k = 1, 2, \cdots$$

where

$$(4.5) \qquad h_1^{(m,a)}(s) = 1 + s, \quad \text{if} \quad m \geqq 2$$
$$= 1 + as, \quad \text{if} \quad m = 1$$

and where

$$(4.6) \qquad a_{i,m} = 1, \quad \text{if} \quad i \leqq m - 1$$
$$= a, \quad \text{if} \quad i > m - 1.$$

The following example illustrates the use of the recursive method to obtain the density of $T_n$. Let $n = 5$ and $m = 2$. According to (4.4)–(4.6)

$$h_4^{(2,a)}(s) = 1 + s + (1+s)as^2 + (1 + s + as^2 + as^3)as^3$$
$$+ (1 + s + as^2 + 2as^3 + as^4 + a^2s^5 + a^2s^6)as^4$$
$$= 1 + s + as^2 + 2as^3 + 2as^4 + (a^2 + a)s^5 + 2a^2s^6 + 2a^2s^7 + a^2s^8$$
$$+ a^3s^9 + a^3s^{10}.$$

Accordingly, the density of $T_5$ is obtained by substituting $a = \theta/(1-\theta)$ in $h_4^{(2,a)}(s)$ and multiplying the coefficients of $s$ by $\frac{1}{2}(1-\theta)^3$. Thus one obtains:

| $t$ | $-10$ | $-8$ | $-6$ | $-4$ | $-2$ | $0$ |
|---|---|---|---|---|---|---|
| $P^{(2,\theta)}[T_5 = t]$ | $\frac{1}{2}(1-\theta)^3$ | $\frac{1}{2}(1-\theta)^3$ | $\frac{1}{2}(1-\theta)^2$ | $\theta(1-\theta)^2$ | $\theta(1-\theta)^2$ | $\frac{1}{2}\theta(1-\theta)$ |

| $t$ | $2$ | $4$ | $6$ | $8$ | $10$ |
|---|---|---|---|---|---|
| $P^{(2,\theta)}[T_5 = t]$ | $\theta^2(1-\theta)$ | $\theta^2(1-\theta)$ | $\frac{1}{2}\theta^2(1-\theta)$ | $\frac{1}{2}\theta^3$ | $\frac{1}{2}\theta^3$ |

TABLE 4.1

*The frequency function $2^{n-1}f_{T_n}^{(0)}(t)$ for the symmetric binomial case $n = 2(1)10$*

| $n$ | $t$ = 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | 1 | | | | | | | | | | | | | | | | | | | | | |
| 3 | | 1 | | 1 | | | | | | | | | | | | | | | | | | | |
| 4 | 2 | | 1 | | 1 | | 1 | | | | | | | | | | | | | | | | |
| 5 | 2 | | 2 | | 2 | | 1 | | 1 | | | | | | | | | | | | | | |
| 6 | | 3 | | 3 | | 3 | | 2 | | 2 | | 1 | | 1 | | 1 | | | | | | | |
| 7 | | 5 | | 5 | | 4 | | 4 | | 4 | | 3 | | 2 | | 2 | | 1 | | 1 | | 1 | |
| 8 | 8 | | 8 | | 8 | | 7 | | 7 | | 6 | | 5 | | 5 | | 4 | | 3 | | 2 | | 2 |
| 9 | 14 | | 13 | | 13 | | 13 | | 12 | | 11 | | 10 | | 9 | | 8 | | 7 | | 6 | | 5 |
| 10 | | 23 | | 23 | | 22 | | 21 | | 21 | | 19 | | 18 | | 17 | | 15 | | 13 | | 12 | |

| $n$ | $t$ = 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | | | | | | | | | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | | | | | | | | | |
| 5 | | | | | | | | | | | | | | | | | | | | | | | |
| 6 | | | | | | | | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | | | | | | | | |
| 8 | | 1 | | 1 | | 1 | | | | | | | | | | | | | | | | | |
| 9 | | 4 | | 3 | | 2 | | 2 | | 1 | | 1 | | 1 | | | | | | | | | |
| 10 | 10 | | 9 | | 8 | | 6 | | 5 | | 4 | | 3 | | 2 | | 2 | | 1 | | 1 | | 1 |

Obviously, when $\theta = \frac{1}{2}$ (no change) the density of $T_n$ is symmetric about the origin. In Table 4.1 we give the density of $T_n$ for $\theta = \frac{1}{2}$ and for $n = 2(1)10$.

(ii) *The exponential case.* Without loss of generality assume that $\theta_0 = 1$; i.e.,

$$f_{X_1}(x; \theta_0) = e^{-x}, \qquad x \geqq 0$$
$$= 0, \qquad x < 0.$$

We derive now the distribution of $T_n = \sum_{i=1}^{n-1} iX_{i+1}$ under the null hypothesis, $H_0$. Let $Y_i = iX_{i+1}$ $(i = 1, \cdots, n-1)$. The distribution law of $Y_i$ is the exponential with intensity $i^{-1}$. The density of $T_n$ is obtained from the density of $T_{n-1}$ by a convolution with the density of $Y_{n-1}$. Accordingly, if $f_n^{(0)}(t)$ denotes the density function of $T_n$ under $H_0$ we have,

$$(4.7) \quad f_n^{(0)}(t) = \int_0^t f_{n-1}^{(0)}(u)[1/(n-1)] \exp[-[1/(n-1)](t-u)]\, du,$$

where $f_1^{(0)}(t) = e^{-t}$, $0 \leqq t \leqq \infty$; $n = 2, 3, \cdots$.

It is simple to prove then, by induction on $n$, that the density function of $T_n$ is,

$$(4.8) \qquad f_n^{(0)}(t) = \sum_{j=1}^{n-1} k_{n-1,j}e^{-(1/j)t}, \qquad 0 \leqq t \leqq \infty$$

TABLE 4.2

*The coefficients $k_{n-1,j}$ of the density of $T_n$ in the exponential case, $\theta_0 = 1$, $n = 2(1)10$*

| $n$ | $j$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 2 | 1.000 | | | | | | | | |
| 3 | −1.000 | 1.000 | | | | | | | |
| 4 | 0.500 | −2.000 | 1.500 | | | | | | |
| 5 | −0.167 | 2.000 | −4.500 | 2.667 | | | | | |
| 6 | 0.042 | −1.333 | 6.750 | −10.667 | 5.208 | | | | |
| 7 | −0.008 | 0.667 | −6.750 | 21.333 | −26.042 | 10.800 | | | |
| 8 | 0.001 | −0.267 | 5.062 | −28.444 | 65.104 | −64.800 | 23.343 | | |
| 9 | 0.000 | 0.089 | −3.038 | 28.444 | −108.507 | 194.400 | −163.401 | 52.013 | |
| 10 | 0.000 | −0.025 | 1.519 | −22.756 | 135.634 | −388.800 | 571.905 | −416.102 | 118.625 |

where the coefficients $k_{n-1,j}$ $(j = 1, \cdots, n-1)$ are determined recursively by the formula,

$$\begin{aligned} &k_{1,1} = 1 \\ (4.9)\quad &k_{v,j} = -k_{v-1,j}/((v/j) - 1), \quad j = 1, 2, \cdots, v-1 \text{ for all } v = 2, 3, \cdots \\ &k_{v,v} = -\textstyle\sum_{j=1}^{v-1} k_{v,j}\,. \end{aligned}$$

We notice also that since $\int_0^\infty f_n^{(0)}(t)\,dt = 1$ for all $n$, one gets the identity, $\sum_{j=1}^{n} jk_{n,j} = 1$ for all $n = 1, 2, \cdots$. The coefficients $k_{n-1,j}$ of the density (4.8) are given in Table 4.2 for $n = 2(1)10$.

We derive now the density function of $T_n$ under the alterative hypothesis. This density function will be denoted by $f_n^{(m,\rho)}(t)$, $m = 1, 2, \cdots, n-1$ and $0 < \rho < \infty$; where $\rho$ is the intensity parameter of the $n - m$ last random variables. Write $T_n = T_m + T_{n-m}^*$, where $T_{n-m}^* = \sum_{i=m}^{n-1} iX_{i+1}$. The density function of $T_m$ is $f_m^{(0)}(t)$. Let $f_{n-m}^*(t; \rho)$ denote the density function of $T_{n-m}^*$.

By a backward induction on $m$ we prove that

$$(4.10)\qquad f_{n-m}^*(t;\rho) = \textstyle\sum_{j=m}^{n-1} k_{n-m,j}^*(\rho) \exp\{-(\rho/j)t\}, \qquad 0 \leqq t \leqq \infty$$

where the coefficients $k_{n-m,j}^*(\rho)$ are obtained by the recursive formula,

$$\begin{aligned} &k_{n-m,m}^*(\rho) = -\textstyle\sum_{j=m+1}^{n-1} jk_{n-m-1,j}^*(\rho)/(j-m) \\ (4.11)\quad &k_{n-m,j}^*(\rho) = jk_{n-m-1,j}^*(\rho)/(j-m), \qquad j = m+1, \cdots, n-1 \\ &k_{1,n-1}^*(\rho) = \rho/(n-1). \end{aligned}$$

Finally, since $T_m$ and $T_{n-m}^*$ are independent, the density of $T_n$ is

$$\begin{aligned} f_{T_n}^{(m,\rho)}(t) &= \textstyle\int_0^t f_m^{(0)}(u) f_{n-m}^{(m,\rho)}(t-u)\,du \\ (4.12)\quad &= \textstyle\sum_{j=1}^{n-1} (j/(j-\rho))k_{n-1,j}^*(\rho)[e^{-(\rho/j)t} - e^{-t}], \qquad \text{if } m = 1 \\ &= \textstyle\sum_{i=1}^{m-1}\sum_{j=m}^{n-1} (ij/(j-i\rho))k_{m-1,i}k_{n-m,j}^*(\rho)[e^{-(\rho/j)t} - e^{-(1/i)t}], \\ &\qquad \text{if } 2 \leqq m \leqq n-1. \end{aligned}$$

The cdf $F_{T_n}^{(m,\rho)}(t)$ is obtained easily from (4.12) by integration.

**5. Exact and approximate determination of the critical level and power function of the test based on $T_n$.** Exact determination of the critical level of the test based on $T_n$ and of its power function is very easy in the case of normally distributed random variables (see H. Chernoff and S. Zacks [1]). This task becomes much more involved when the observed random variables are not normally distributed. The question is whether one can obtain fairly good results by using normal approximation even in small sample situations (samples of size $n = 5$ or $n = 10$)? In the present section we perform several numerical comparisons of the exact and approximate power functions for the binomial and exponential cases.

5.1. *The binomial case.* Suppose we want to test the hypothesis $H_0 : \theta_1 = \cdots = \theta_n = \frac{1}{2}$ against the one-sided alternative $H_1 : \theta_1 = \cdots = \theta_m = \frac{1}{2}, \theta_{m+1} = \cdots = \theta_n = \frac{1}{2} + \delta, 0 < \delta < \frac{1}{2}$ ($m$ unknown). The exact test criterion for the procedure based on $T_n$, for a given level of significance, $\alpha$, can be found from the exact density of $T_n$. For values of $n \leqq 10$ we can get it directly from Table 4.1. For example, if $n = 10$, and $\alpha = 0.05$ the hypothesis $H_0$ is always rejected if $T_{10} \geqq 29$ and rejected with probability 0.075 if $T_{10} = 27$. If $T_{10} \leqq 25$ the hypothesis $H_0$ is not rejected. If $\alpha = 0.01$, $H_0$ is rejected whenever $T_{10} \geqq 39$; rejected with probability 0.06 if $T_{10} = 37$ and is not rejected if $T_{10} \leqq 35$. Denote by $C_\alpha$ the value for which $H_0$ is rejected w.p.1 if $T_n > C_\alpha$, and let $\gamma_\alpha$ be the probability of rejecting $H_0$ if $T_n = C_\alpha$. Then, the power of the test is

$$\beta_{n,m}(\delta) = \gamma_\alpha P^{(m,\delta)}[T_n = C_\alpha] + \sum_{t>C_\alpha} P^{(m,\delta)}[T_n = t] \tag{5.1}$$

where $P^{(m,\delta)}[T_n = t]$ is the density of $T_n$ under $H_1$.

A normal approximation to $C_\alpha$ is obtained by solving the equation

$$\alpha = 1 - \Phi(\tilde{C}_\alpha/(n(n-1)(2n-1)/6)^{\frac{1}{2}}) \tag{5.2}$$

where $\tilde{C}_\alpha$ denotes the approximate critical level. Whenever $T_n \geqq \tilde{C}_\alpha$ the hypothesis $H_0$ is rejected. According to (5.2) the approximate critical level is,

$$\tilde{C}_\alpha = U_{1-\alpha}[n(n-1)(2n-1)/6]^{\frac{1}{2}} \tag{5.3}$$

where $U_{1-\alpha}$ denotes the $(1 - \alpha)$th fractile of the standard normal random variable. If one uses the critical level $\tilde{C}_\alpha$ the level of significance actually attained

TABLE 5.1
*Approximate critical level $\tilde{C}_\alpha$ and attained level of significance $\tilde{\alpha}$ in the binomial case, $n = 5, 10$*

| | $n$ | $\alpha$ | | | |
|---|---|---|---|---|---|
| | | 0.01 | 0.025 | 0.05 | 0.10 |
| $\tilde{\alpha}$ | 5 | 0 | 0 | 0.063 | 0.125 |
| | 10 | 0.006 | 0.020 | 0.049 | 0.102 |
| $\tilde{C}_\alpha$ | 5 | 12.7 | 10.7 | 9.0 | 7.0 |
| | 10 | 39.3 | 33.1 | 27.8 | 21.6 |

TABLE 5.2

*The exact power function $\beta_{n,m}(\theta)$ and its normal approximation $\tilde{\beta}_{n,m}(\theta)$ for the binomial case, $n = 10$*

| $m$ | $\theta$ = 0.6, $\alpha = 0.01$ | $\theta$ = 0.6, $\alpha = 0.05$ | $\theta$ = 0.7, $\alpha = 0.01$ | $\theta$ = 0.7, $\alpha = 0.05$ | $\theta$ = 0.8, $\alpha = 0.01$ | $\theta$ = 0.8, $\alpha = 0.05$ | $\theta$ = 0.9, $\alpha = 0.01$ | $\theta$ = 0.9, $\alpha = 0.05$ | |
|---|---|---|---|---|---|---|---|---|---|
| 0–1 | 0.0354 | 0.1320 | 0.1011 | 0.2846 | 0.2458 | 0.5172 | 0.5242 | 0.7960 | exact |
| | 0.0336 | 0.1281 | 0.0845 | 0.2640 | 0.1820 | 0.4773 | 0.3738 | 0.7916 | approx. |
| 2 | 0.0343 | 0.1293 | 0.0962 | 0.2751 | 0.3232 | 0.4981 | 0.5050 | 0.7724 | exact |
| | 0.0328 | 0.1257 | 0.0807 | 0.2555 | 0.1706 | 0.4598 | 0.3446 | 0.7676 | approx. |
| 4 | 0.0298 | 0.1150 | 0.0748 | 0.2282 | 0.1663 | 0.4059 | 0.3366 | 0.6608 | exact |
| | 0.0287 | 0.1139 | 0.0640 | 0.2168 | 0.1232 | 0.3749 | 0.2224 | 0.6274 | approx. |
| 6 | 0.0207 | 0.0951 | 0.0384 | 0.1647 | 0.0655 | 0.2665 | 0.1050 | 0.4087 | exact |
| | 0.0227 | 0.0949 | 0.0417 | 0.1585 | 0.0674 | 0.2461 | 0.0968 | 0.3741 | approx. |
| 8 | 0.0143 | 0.0709 | 0.0196 | 0.0955 | 0.0256 | 0.1237 | 0.0324 | 0.1555 | exact |
| | 0.0159 | 0.0723 | 0.0224 | 0.0975 | 0.0284 | 0.1247 | 0.0320 | 0.1539 | approx. |

is not $\alpha$ but some $\tilde{\alpha}$ close to $\alpha$. In Table 5.1 the values of $\tilde{\alpha}$ and $\tilde{C}_\alpha$ for some values of $\alpha$, and samples of size $n = 5, 10$ are given. The values $\tilde{\alpha}$ were determined by the exact distribution of $T_n$ under $H_0$.

We see that the normal approximation is poor for a sample of size as small as $n = 5$, but is sufficiently good for sample of size $n = 10$ (compare with exact values for $n = 10$ and $\alpha = 0.01, 0.05$ which are given above).

The normal approximation to the power function $\beta_{n,m}(\delta)$ is given by,

$$(5.4)\quad \tilde{\beta}_{n,m}(\theta) = 1 - \Phi\left[\frac{U_{1-\alpha} - (2\theta - 1)\left(\frac{3n(n-1)}{2(2n-1)}\right)^{\frac{1}{2}}\left(1 - \frac{m(m-1)}{n(n-1)}\right)}{\left[4\theta(1-\theta) + \frac{m(m-1)(2m-1)}{n(n-1)(2n-1)}(1 - 4\theta(1-\theta))\right]^{\frac{1}{2}}}\right]$$

where $\theta = \frac{1}{2} + \delta$. In Table 5.2 we compare the exact power function and the approximate one for a sample of size $n = 10$, and levels of significance $\alpha = 0.05$, $0.10$. As seen in this table, the normal approximation is quite good in all the range of $\theta$ values when the level of significance is $\alpha = 0.05$. When $\alpha$ is 0.01 the normal approximation is not effective for large values of $\theta$ and small $m$'s (see for example $m = 2$ and $\theta = 0.9$). In such cases we can try to improve by adding to (5.4) an extra term from expansion (3.5).

5.2. *The exponential case.* The exact critical level $C_\alpha$ for rejecting the null hypothesis $H_0$ in the exponential case is the root of the equation

$$(5.5)\qquad \textstyle\sum_{j=1}^{n-1} jk_{n-1,j}e^{-C_\alpha/j} = \alpha$$

where $k_{n-1,j}$ are the coefficients of $e^{-t/j}$ in Expression (4.8), recursively defined in (4.9). Here we test the hypothesis that $\theta_0 = 1$ against the alternative that after the $m$th observation $\theta = \rho < 1$ (the expected value of $X$ is larger than 1). $H_0$

is rejected whenever $T_n > C_\alpha$. $C_\alpha$ is the $(1 - \alpha)$th fractile of the distribution of $T_n$ under $H_0$. If one wishes to test $H_0$ against the alternative that $\rho > 1$ then the critical level will be the $\alpha$th fractile of the distribution of $T_n$, say $C_\alpha'$, and $H_0$ is rejected whenever $T_n \leqq C_\alpha'$. Two sided critical levels can similarly be found for testing against a two sided alternative. A graphical solution of Equation (5.5) can be used. This will always yield a satisfactory degree of accuracy. We shall consider here a normal approximation to $C_\alpha$ and a Newton-Raphson one cycle correction of the normal approximation.

It is easy to verify that the normal approximation to the critical level of $T_n$ (testing against $\rho < 1$) is,

$$(5.6) \qquad \tilde{C}_\alpha = \tfrac{1}{2}n(n-1) + U_{1-\alpha}[\tfrac{1}{6}n(n-1)(2n-1)]^{\frac{1}{2}}.$$

This approximation yields, in samples of size $n = 5$ and 10, the following actual levels of significance, $\tilde{\alpha}$ (determined according to (5.5)).

TABLE 5.3

*Levels of significance $\tilde{\alpha}$ attained by the normal approximation in the exponential case*

| | $n$ | $\alpha$ | | | |
|---|---|---|---|---|---|
| | | 0.01 | 0.025 | 0.05 | 0.10 |
| $\tilde{\alpha}$ | 5 | 0.0297 | 0.0454 | 0.0686 | 0.1065 |
| | 10 | 0.0235 | 0.0458 | 0.0730 | 0.1397 |
| $\tilde{C}_\alpha$ | 5 | 22.7 | 20.7 | 19.0 | 17.0 |
| | 10 | 84.3 | 78.1 | 72.8 | 66.6 |

As Table 5.3 shows, the normal approximation (5.6) to the critical level of the test in the exponential case is unsatisfactory. There is not much difference in the goodness of the normal approximation between samples of size $n = 5$ or $n = 10$. The value of $\tilde{C}_\alpha$ obtained by (5.6) can be used as initial solution of (5.5) and then corrected according to the Newton-Raphson's method. A one-cycle correction of $\tilde{C}_\alpha$ leads to the formula:

$$(5.7) \quad C_\alpha^* = \tilde{C}_\alpha + [\textstyle\sum_{j=1}^{n-1} jk_{n-1,j} \exp\{-\tilde{C}_\alpha/j\} - \alpha] \cdot [\textstyle\sum_{j=1}^{n-1} k_{n-1,j} \exp\{-\tilde{C}_\alpha/j\}]^{-1}.$$

In a two cycle approximation the value of $C_\alpha^*$ will be substituted in (5.7) for $\tilde{C}_\alpha$ to yield $C_\alpha^{**}$, etc. In most applications a one cycle correction of $\tilde{C}_\alpha$ will be sufficient. In the following table we present a one cycle correction of the approximation given in Table 5.3.

As seen in Table 5.4 the one-cycle Newton-Raphson's correction of the normal approximation is very effective, even in samples of size $n = 5$. We propose, therefore, that the critical levels will be determined, in the exponential case, according to Formulae (5.6) and (5.7).

TABLE 5.4

*Levels of significance $\alpha^*$ and critical levels $C_\alpha^*$ attained by a Newton-Raphson correction of the normal approximation in the exponential case*

| | $n$ | $\alpha$ | | | |
|---|---|---|---|---|---|
| | | 0.010 | 0.025 | 0.050 | 0.100 |
| $\alpha^*$ | 5 | 0.0152 | 0.0292 | 0.0522 | 0.1002 |
| | 10 | 0.0135 | 0.0279 | 0.0516 | 0.1002 |
| $C_\alpha^*$ | 5 | 25.57 | 22.76 | 20.22 | 17.27 |
| | 10 | 91.09 | 82.94 | 75.73 | 67.45 |

The exact power function, $\beta_{n,m}(\rho)$, when the critical level is $C_\alpha$ is given, according to (4.21), by

$$\begin{aligned}\beta_{n,m}(\rho) &= P^{(m,\rho)}[T_n \geqq C_\alpha^*] \\ &= \sum_{i=1}^{m-1} ik_{m-1,i} \sum_{j=m}^{n-1} [j^2/(j - i\rho)\rho]k^*_{n-m,j}(\rho) \exp\{-(\rho/j)C_\alpha^*\} \\ &\quad - \sum_{i=1}^{m-1} i^2 k_{m-1,i} \exp\{-(1/i)C_\alpha^*\} \sum_{j=m}^{n-1} [j/(j - i\rho)]k^*_{n-m,j}(\rho)\end{aligned} \tag{5.8}$$

whenever $m \geqq 2$; and a simpler expression is found for the case of $m = 1$. The computation of $\beta_{n,m}(\rho)$ according to (5.8), might often be too difficult and time consuming. Since $\beta_{n,m}(\rho) = 1 - F_{T_n}^{(m,\rho)}(C_\alpha^*)$ we can approximate this power function according to (3.5). The expectation and variance of $T_n$ under $H_1$ are:

$$E\{T_n \mid m, \rho\} = (1/\rho)[\tfrac{1}{2}n(n - 1) - (1 - \rho)\tfrac{1}{2}m(m - 1)] \tag{5.9}$$

and

$$\operatorname{Var}\{T_n \mid m, \rho\} = (1/\rho^2)[\tfrac{1}{6}n(n - 1)(2n - 1) - (1 - \rho^2)\tfrac{1}{6}m(m - 1)(2m - 1)]. \tag{5.10}$$

Using semi-invariants one can easily prove that the coefficients of asymmetry and kurtosis of $T_n$ under $H_1$ are:

$$\gamma_1(T_n \mid m, \rho) = \frac{3(6)^{\frac{1}{2}}[(n - 1)^2n^2 - (1 - \rho^3)(m - 1)^2m^2]}{[n(n - 1)(2n - 1) - (1 - \rho^2)m(m - 1)(2m - 1)]^{\frac{3}{2}}} \tag{5.11}$$

and

TABLE 5.5

*Power function of $T_5$ in the exponential case, for $n = 5$, $m = 2$, $\alpha = 0.05$*

| $\rho$ | Power function | | |
|---|---|---|---|
| | exact | approximation (5.13) | normal approximation |
| 0.8 | 0.1162 | 0.1377 | 0.1196 |
| 0.6 | 0.2770 | 0.2864 | 0.3188 |
| 0.4 | 0.4448 | 0.5038 | 0.5948 |
| 0.2 | 0.6068 | 0.7964 | 0.8305 |

$$(5.12)\quad \gamma_2(T_n \mid m, \rho) = 10.8\,\frac{n(n-1)(2n-1)[3n(n-1)-1] - (1-\rho^4)m(m-1)(2m-1)[3m(m-1)-1]}{[n(n-1)(2n-1)-(1-\rho^2)m(m-1)(2m-1)]^2} - 3.$$

Substituting (5.9)–(5.12) in (3.5) we obtain the following approximation for $F_{T_n}^{(m,\rho)}(t)$,

$$(5.13)\quad \begin{aligned} F_{T_n}^{(m,\rho)}(t) \cong \Phi\left(\frac{t - E\{T_n \mid m, \rho\}}{[\text{Var}\,\{T_n \mid m, \rho\}]^{\frac{1}{2}}}\right) &- \frac{1}{6}\gamma_1(T_n \mid m, \rho)\cdot\Phi^{(3)}\left(\frac{t - E\{T_n \mid m, \rho\}}{[\text{Var}\,\{T_n \mid m, \rho\}]^{\frac{1}{2}}}\right) \\ &+ \frac{1}{24}\gamma_2(T_n \mid m, \rho)\Phi^{(4)}\left(\frac{t - E\{T_n \mid m, \rho\}}{[\text{Var}\,\{T_n \mid m, \rho\}]^{\frac{1}{2}}}\right) \\ &+ \frac{1}{72}\gamma_1^2(T_n \mid m, \rho)\Phi^{(6)}\left(\frac{t - E\{T_n \mid m, \rho\}}{[\text{Var}\,\{T_n \mid m, \rho\}]^{\frac{1}{2}}}\right). \end{aligned}$$

In Table 5.5 we compare the exact power function of $T_n$ in the exponential case, to the normal approximation and to the Approximation (5.13). In this example we treat a very small sample size, $n = 5$.

As seen in the present table Approximation (5.13) yields results which are generally closer to the exact values of the power function than those obtained by the normal approximation. We expect that in samples of size $n \geqq 10$ the Approximation (5.13) will be very effective. In situations of very small sample size it is not difficult to determine the exact values of the power function $\beta_{n,m}(\rho)$ from Formula (5.8).

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