NOTE ON A THEOREM OF KINGMAN AND A THEOREM OF CHUNG

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Let $P = \{p_{ij}\}$ be the matrix of transition probabilities of an irreducible, aperiodic Markov chain. It is known (see [3]) that if the chain is transient, the iterated probabilities $\{p_{ij}^{(n)}\}$ may tend geometrically to zero, in which case there is a common value $R > 1$ such that, for all $i, j$, $\{p_{ij}^{(n)}R^n\}$ tends to a finite limit as $n \to \infty$, but $\{p_{ij}^{(n)}r^n\}$ is divergent for $r > R$. Kingman [2] has called this the case of "geometric transience," and shown that, under the conditions below, if $\{u_i\}$ is an initial distribution, and $C$ some set of states, the quantities

$$P_j(n) = \sum u_i p_{ij}^{(n)}$$

and

$$Q_C(n) = \sum_{i \in C} p_{ij}^{(n)}$$

satisfy $\lim_{n \to \infty} [P_j(n)]^{1/n} = \lim_{n \to \infty} [Q_C(n)]^{1/n} = 1/R$.

(Kingman discusses a continuous time process, but his results apply with the obvious changes in the present context.)

The conditions to be satisfied by $\{u_i\}$ and $C$ are stated in terms of solutions to the inequalities

$$R \sum p_{ij} \beta_j \leq \beta_i \quad (\beta_i > 0),$$

$$R \sum \alpha_j p_{ij} \leq \alpha_i \quad (\alpha_j > 0).$$

(It can be shown that non-trivial solutions to these inequalities always exist. We shall call them right and left $R$-subinvariant vectors respectively.) Kingman's condition on the vector $\{u_i\}$ is that it should satisfy the condition $\sum u_i \beta_i < \infty$ for some right $R$-subinvariant vector $\{\beta_i\}$, and the condition on the set of states $C$ is that it should satisfy the condition $\sum_{i \in C} \alpha_i < \infty$ for some left $R$-subinvariant vector $\{\alpha_i\}$.

The purpose of this note is to use the general theory developed in [4] to show that in fact these conditions imply a stronger result, namely the convergence of the quantities $P_j(n)R^n$ and $Q_C(n)R^n$ to finite limits. We shall also apply the results of [4] to two theorems of Chung's concerning the convergence of functionals of a Markov chain.

The discussion in [4] concerns the convergence of the more general sums

$$P_j(n; R) = \sum_i u_i p_{ij}^{(n)}R^n;$$

$$Q_i(n; R) = \sum_i t_i^{(n)} v_j R^n;$$

$$S(n; R) = \sum \sum u_i t_i^{(n)} v_j R^n;$$

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where $T$ is any (not necessarily stochastic) irreducible, aperiodic, non-negative matrix with convergence parameter $R$, and $\{u_i\}, \{v_i\}$ are any (not necessarily non-negative) vectors. As before, it can be shown that there always exist positive left and right $R$-subinvariant vectors, say $\{\alpha_i\}, \{\beta_i\}$, while suitable conditions on the vectors $\{u_i\}$ and $\{v_i\}$ take the form

\begin{align*}
(3) & \quad \sum |u_i| \beta_i < \infty \quad \text{for some right } R\text{-subinvariant vector } \{\beta_i\}, \\
(4) & \quad \sum \alpha_i |v_i| < \infty \quad \text{for some left } R\text{-subinvariant vector } \{\alpha_i\}.
\end{align*}

Then it is proved that

(A) (3) is a sufficient condition for the convergence to a finite limit of the sequence $P_t(n; R) \ (n \to \infty)$;

(B) (4) is a sufficient condition for the convergence to a finite limit of the sequence $Q_c(n; R)$;

(C) (3), (4), and the supplementary condition either $|u_i| \leq K \alpha_i$ for some $K < \infty$, or $|v_i| \leq K' \beta_i$ for some $K' < \infty$, are sufficient to ensure the convergence to a finite limit of the sequence $S(n; R)$.

When the appropriate conditions are satisfied, the limits can be computed by interchanging the limit and summation operations, i.e. they are zero whenever the matrix is $R$-transient or $R$-null, and equal respectively to

\begin{align*}
\left( \sum u_i \beta_k / \sum \alpha_k \beta_k \right) \alpha_j, \quad \sum \alpha_k v_k / \sum \alpha_k \beta_k \beta_i, \quad \left( \sum \alpha_k \beta_k / \sum \alpha_k \beta_k \right)^2 \left( \sum \alpha_k \beta_k / \sum \alpha_k \beta_k \right) \alpha_j
\end{align*}

when the matrix is $R$-positive, (when the vectors $\{\alpha_i\}$ and $\{\beta_i\}$ are uniquely defined (up to constant factors), strictly invariant, and satisfies the condition $\sum \alpha_k \beta_k < \infty$).²

Applying these results to Kingman’s problem, we see that under his conditions, the sums $P_t(n) R^n$ and $Q_c(n) R^n$ tend to finite limits which are zero if $P$ is $R$-transient or $R$-null, and equal to $\left( \sum_k \pi_k \beta_k / \sum_k \alpha_k \beta_k \right) \alpha_j$ and $\left( \sum_k \pi_k \alpha_k / \sum \alpha_k \beta_k \right) \beta_i$ respectively if $P$ is $R$-positive.

As a second application, suppose that the chain is positive recurrent, and let \{z_n\} denote the sequence of random variables whose transition matrix is described by $P$. Then as $n \to \infty$, the distribution of $z_n$ tends to a limit $\{\pi_i\}$ which is a left invariant vector for $P$. If $f(\cdot)$ is any function from the state space onto the real, we shall call the sequence $y_n = f(z_n)$ a functional of the Markov chain. Applying criterion (B) for the matrix $P$, with $R = 1$, $\alpha_k = \pi_k$ and $v_k = f(k)^r$ $(r > 0)$, and supposing that initially $z_0 = i$, we obtain the following theorem for the convergence of the moments of $y_n$.

**Theorem.** Let $\{z_n\}$, $f(\cdot)\{\pi_i\}$ be defined as above. Then the moments $E(y_n^r)$ exist and tend to a finite limit if the corresponding absolute moment of the limit distribution, $E[y_\infty^r] = \sum \pi_i |f(j)|^r$, is finite, in which case $E(y_n^r) \to E(y_\infty^r)$.

It is not difficult to show that the conditions (A) and (B) are necessary as well

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¹ The terminology is that used in [3] and [4], to which the reader is referred for further explanation and a proof of these results.

² See footnote number 1.
as sufficient for the convergence of the sums $P_j(n; R)$, $Q_j(n, R)$ if the matrix is $R$-positive and the vectors are non-negative; hence the condition of the theorem is necessary if the function $f(\cdot)$ is non-negative.

By making use of (C) it is possible to extend the results to the case when the initial distribution is not restricted to a single state. For example, the conclusions of the theorem will continue to hold if, in addition to the condition $E[y_0]^r < \infty$, the initial distribution is dominated by some multiple of the limit distribution.

These results are only a slight extension of those of Chung [1], Theorems I. 14.5 and I. 15.4; the main point of our discussion is that it shows that the analytical content of the theorems can be obtained very readily by direct arguments.

REFERENCES