MONOTONE CONVERGENCE OF MOMENTS IN AGE DEPENDENT BRANCHING PROCESSES

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1. Introduction. Let $Z(t)$ be the number of cells at time t of a branching process starting at $t = 0$ with one new cell. Let $N(t)$ be the total number of such cells born by time t. Each cell has lifetime distribution function $G(t)$ with $G(0) = 0$. At the end of its life the cell disappears and is replaced by k cells with probability p_k, $k = 0, 1, 2, \cdots$, where $p_k \geqq 0$ and $\sum_{k=0}^{\infty} p_k = 1$. Each cell has lifetime distribution function $G(t)$ and proceeds independently of the state of the system, and identically as the parent cell. Such a process, for this general G, is called an age-dependent branching process and is extensively treated in [1].

Define $h(s) = \sum_{k=0}^{\infty} p_k s^k$. For $h'(1) \equiv m > 1$, there is an increasing population with probability one. For this case, by use of Smith's key renewal theorem, it is shown ([1], Ch. 6) that for $\int_0^{\infty} u\, dG(u) < \infty$, as $t \to \infty$,

$$E[Z(t)] \sim K_1 \exp(\alpha t) \tag{1}$$

$$E[N(t)] \sim K_2 \exp(\alpha t) \tag{2}$$

where K_1, K_2 and α man be evaluated.

Necessary and sufficient conditions for the monotone convergence of $E[Z(t)\exp(-\alpha t)$ and $E[N(t)]\exp(-\alpha t)$ were given in [2].

It is the purpose of this note to show that for $m > 1$ that

$$M_n(t)\exp(-n\alpha t) \uparrow d_n < \infty, \quad \text{for} \quad n \geqq 2, \tag{3}$$

where $M_n(t)$ is the nth factorial moment of $Z(t)$. This will be done by exhibiting the monotone nature of the solution of an integral equation in Section 2, a special case of which determines the corresponding monotone behavior in (3), as is given in Section 3.

2. Integral equation. A special case of the following theorem was given in [2].

Theorem. *Let $Q(t)$, defined on the positive axis, satisfy the following equation*

$$Q(t) = \int_0^t [(f(t-u) + b(u))Q(t-u) + k(u) + l(t-u)]g(u)\, du \tag{4}$$

where f, b, k, l are non-negative functions defined on the positive axis with f and l non-decreasing, $f + b \leqq 1$, and g is a probability density strictly positive on the entire positive axis. Then $Q(t)$ is non-decreasing.

Proof. Let $\{X_i\}_1^{\infty}$ be independent and identically distributed random variables on the positive axis, each with density $g(u)$. Let $S_n = \sum_{i=1}^{n} X_i$, $n \geqq 1$. Let $Y_i = k(X_i)$, $Z_i = l(t - S_i)$ and let $N(t) = \max\{n : S_n \leqq t\}$.

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Let

(5) $$W(t) \equiv \sum_{i=1}^{N(t)} (Y_i + Z_i) \prod_{j=1}^{i-1} (f(t - S_j) + b(X_j))$$ and

(6) $$Q_0(t) \equiv E[W(t)],$$

where $\sum_{i=1}^{0} = 0$ and $\prod_{j=1}^{0} = 1$ by convention

Then one may write, following a suggestion of Prof. R. Pyke,

(7) $$W(t) = Y_1 + Z_1 + W_1(t - X_1)[f(t - X_1) + b(X_1)]$$

where W and W_1 are independent and identically distributed. From (7) it is clear that (6) is a solution to (4).

To show uniqueness, let $h(t) \equiv |Q_0(t) - Q_1(t)|$ be the absolute difference of two solutions to (4). Then clearly $h(t) \leqq \int_0^t h(t-u)g(u)\,du$. Suppose $0 < a_l = \sup_{0 \leqq t \leqq l} h(t)$. Then it follows that $a_l \leqq a_l G(l) < a_l$, a contradiction. Hence (6) is the unique solution to (4) and the result follows.

3. Monotonicity of normalized moments.

THEOREM. *Let $m > 1$ in a simple age-dependent branching process defined in the introduction. Then, defining $\alpha > 0$ by $m \int_0^\infty \exp(-\alpha u)g(u)\,du = 1$, where $g(u) = G'(u)$ exists* a.e., *it follows that $M_n(t) \exp(-n\alpha t)$ is monotone increasing to a finite limit as $t \to \infty$ for all $n \geqq 1$ if in addition,*

$$g(u) > \alpha[1 - G(u)][m-1]^{-1} \quad \textit{for} \quad u \geqq 0.$$

PROOF. The theorem holds for $n = 1$ by the theorem in Section 3 of [2]. Let $F(x, t) = E[s^{Z(t)}]$. Then ([1], Ch. 6)

$$F(s,t) = s(1 - G(t)) + \int_0^t h(F(s, t-u))g(u)\,du.$$

By successive differentiation, for $n \geqq 2$, $M_n(t)\exp(-n\alpha t) \equiv M_n^*(t)$ satisfies an equation of the form

$$M_n^*(t) = \int_0^t [a_n M_n^*(t-u) + k_n(t-u)]g_n^*(u)\,du$$

where $g_n^*(u)$ is a density (different from g), and where $0 < a_n = m \int_0^\infty \exp(-\alpha u n)g(u)\,du < 1$ and k_n is a positive, increasing function. The theorem of Section 2 establishes the monotonicity.

By induction and use of approximations by writing $\int_0^t k_n(t-u)g_n^*(u)\,du = \int_0^{t\gamma} + \int_{t\gamma}^t k_n(t-u)g_n^*(u)\,du$ for some $0 < \gamma < 1$, it is established that $\lim_{t\to\infty} \int_0^t k_n(t-u)g_n^*(u)\,du \uparrow c_n < \infty$, since by [2] and the integral equation for $M_2(t)$, it is readily seen to hold for $n = 1, 2$.

That $M_n(t)\exp(-n\alpha t) \uparrow d_n < \infty$ for $n \geqq 2$ now follows by standard Abelian and Tauberian theorems on p. 182 and p. 192 of [3], respectively, or directly by Lemma 4, pp. 161, 163 of [1].

4. Remarks. By differentiation of

$$E[\exp(sZ(t))] \equiv R(s,t) = \exp(s)(1 - G(t)) + \int_0^t h(R(s, t-u))g(u)\,du$$

a sufficient condition that $E[Z^n(t)] \exp(-n\alpha t)$ be monotone increasing for $n \geqq 2$ may be obtained similarly, although it will be of more complex form than the simple condition given in Section 3 for the factorial moments. It would be of interest to determine when the condition in Section 3 would suffice for the monotone increase of the normalized moments $E[Z^n(t)] \exp(-n\alpha t)$, $n \geqq 2$.

REFERENCES

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