TOLERANCE AND CONFIDENCE LIMITS FOR CLASSES OF DISTRIBUTIONS BASED ON FAILURE RATE

By Richard E. Barlow[1] and Frank Proschan

University of California, Berkeley, and Boeing Scientific Research Laboratories

1. Introduction. A fundamental problem in statistical reliability theory and life testing is to obtain lower tolerance limits as a function of sample data, say $\underline{X} = (X_1, X_2, \cdots, X_n)$. That is, if X denotes the time to failure of an item with distribution F, then we seek a function $L(\underline{X})$ such that

$$P_F\{1 - F[L(\underline{X})] \geqq 1 - q\} \geqq 1 - \alpha.$$

We call $1 - q$ the population coverage for the interval $[L(\underline{X}), \infty]$ and $1 - \alpha$ the confidence coefficient. Also, we want $U(\underline{X})$ such that $P_F\{F[U(\underline{X})] \geqq q\} \geqq 1 - \alpha$. Related problems are those of obtaining confidence limits on moments and percentiles.

Parametric tolerance limits based on the normal and exponential distributions are well known [8], [9], [14]. Goodman and Madansky [10] examine various criteria for goodness of tolerance intervals and certain optimum properties of the usual exponential tolerance limits are demonstrated. Recently, a great deal of effort has been devoted to obtaining various confidence limits for the Weibull distribution. Dubey [6] obtains asymptotic confidence limits on $1 - F(T)$ and the failure rate for the class of Weibull distributions with nondecreasing failure rate. He also studies the properties of various estimators for Weibull parameters, [7]. Johns and Liberman [12] present a method for obtaining exact lower confidence limits for $1 - F(T)$ when F is the Weibull distribution with both scale and shape parameters unknown. Unlike Dubey, they do not require that the Weibull distribution in question have a nondecreasing failure rate. These confidence limits are obtained both for the censored and noncensored cases and are asymptotically efficient.

There exist distribution-free tolerance limits, [13], based on say, the kth order statistic X_k for certain values of q, α, k and sample size N. However, they have one unfortunate disadvantage. For given α, q, k there is a minimum sample size $N(\alpha, q, k)$ such that

$$P_F\{1 - F(X_k) \geqq 1 - q\} \geqq 1 - \alpha$$

is true only if $N \geqq N(\alpha, q, k)$. Hanson and Koopmans [11] obtain upper tolerance limits for the class of distributions with increasing hazard rate and lower

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tolerance limits for the class of distributions for which log F is concave. They do not assume non-negative random variables as we do.

Assuming that the sample data arise from a distribution with monotone failure rate (either nondecreasing or nonincreasing and $F(0^-) = 0$) or with monotone failure rate average, we obtain conservative confidence limits for most reliability parameters of interest. (See [2] Chapter 2 and Appendix 2 for a discussion of such distributions and a test for monotone failure rate.) These confidence limits are, in part, derived as in the case of the exponential distribution. In many cases, these are optimum confidence limits when the failure distribution is actually exponential [10]. They also have the advantage that they are convenient to compute and are *not* based on a strong, nonverifiable, parametric assumption.

It is important to note that the conservative tolerance limits we obtain are of greatest value when the sample size is small enough so that the distribution-free tolerance limits do not exist. On the other hand, for very large sample size, the distribution-free tolerance limit is close to the percentile providing the coverage desired, whereas the conservative tolerance limit we obtain is, in general, not. A study is under way comparing the distribution-free tolerance limits, the Hanson-Koopmans tolerance limits, and our conservative tolerance limits for various underlying distributions and sample sizes.

Preliminaries. Throughout this paper we use the following notation and assumptions. Let $0 \leqq X_1 \leqq X_2 \leqq \cdots \leqq X_n$ $(0 \leqq Y_1 \leqq Y_2 \leqq \cdots \leqq Y_n)$ denote an ordered sample from a distribution $F(G)$ and define $X_0 = Y_0 = 0$. We assume that F is continuous, $F(0) = G(0) = 0$ and let $G(x) = 1 - e^{-x}$ for $x \geqq 0$. We say that a distribution F with density f is an increasing failure rate (IFR) distribution if its failure rate $r(t) = f(t)/[1 - F(t)]$ is increasing. It is easy to verify that if F is IFR, $G^{-1}F(t) = -\ln [1 - F(t)]$ is convex where finite. This motivates the more general definition: We say that F is IFR if $-\ln [1 - F(t)]$ is convex where finite. Similarly, F is a decreasing failure rate (DFR) distribution if $G^{-1}F(t)$ is concave on $[0, \infty)$. Barlow and Proschan [3], [4] obtain inequalities for expected values of statistics based on the exponential assumption when in fact the true distribution has a monotone failure rate.

We will also be interested in a considerably weaker restriction on F. If F has a density f and failure rate $r(x)$ such that $t^{-1}\int_0^t r(x)\,dx$ is increasing (decreasing) in t, we say that F has an increasing (decreasing) failure rate average. We write F is IFRA (DFRA). More generally, F is IFRA (DFRA) if and only if $G^{-1}F(x)/x \equiv -\ln[1 - F(x)]/x$ is increasing where finite (decreasing on $[0, \infty)$). See [5] for additional properties of this class. If F is IFR (DFR) and $F(0) = 0$, then it follows that F is IFRA (DFRA).

Perhaps a simple example will serve to motivate the IFRA class of distributions. Let

$$\begin{aligned} F(x) &= 0, & x &< 0 \\ &= (1 - e^{-x})(1 - e^{-kx}), & x &\geqq 0 \end{aligned}$$

where $k > 1$. Then it is easy to check that F is IFRA but *not* IFR. This is the life distribution of a structure composed of two substructures in parallel, the first having k components in series, the second consisting of a single component, with component life lengths independently distributed according to the unit exponential distribution. Any "reasonable" structure built from components having exponential or IFR failure distributions will have an IFRA failure distribution (cf. [5]).

We say X_1 is stochastically smaller than X_2 (written $X_1 \leqq_{st} X_2$) if $F_1(x) \geqq F_2(x)$ for all x, where F_1 is the distribution of X_1 and F_2 is the distribution of X_2. We say X_1 is stochastically equal to X_2 (written $X_1 =_{st} X_2$) if X_1 and X_2 have the same distribution.

2. Lower confidence limits. Let

$$\hat{\theta}_{r,n}(\underset{\sim}{X}) = \textstyle\sum_1^r (n - i + 1)r^{-1}(X_i - X_{i-1}),$$

let $\chi^2_{1-\alpha}(2r)$ denote the $(1 - \alpha)$ 100 per cent point of a chi-square distribution with $2r$ degrees of freedom, and let $B_{\alpha,q,r} = -2r \ln (1 - q)/\chi_\alpha^2(2r)$. If

$$L(\underset{\sim}{X}) = B_{1-\alpha,q,r}\hat{\theta}_{r,n}(\underset{\sim}{X}),$$

then

$$P_G\{1 - G[L(\underset{\sim}{Y})] \geqq 1 - q\} = 1 - \alpha.$$

See [9] for this result and tables. Also define $C_{1-\alpha,q,r} = \min (B_{1-\alpha,q,r}, r/n)$.

THEOREM 2.1. *If F is* IFR, $F(0) = 0$, $F(\xi_q) = q$, *then*

$$P_F\{1 - F[C_{1-\alpha,q,r}\hat{\theta}_{r,n}] \geqq 1 - q\} \geqq 1 - \alpha, \tag{2.1}$$

or equivalently,

$$P_F\{\xi_q \geqq C_{1-\alpha,q,r}\hat{\theta}_{r,n}\} \geqq 1 - \alpha. \tag{2.2}$$

PROOF. Since (2.1) and (2.2) are equivalent, we need only show (2.1). Note that $\sum_1^r a_iX_i = \sum_1^r \bar{A}_i(X_i - X_{i-1})$ where $\bar{A}_i = \sum_{j=i}^r a_j$, as in [3]. By Theorem 4.2 of [3],

$$F[\textstyle\sum_1^r \bar{A}_i(X_i - X_{i-1})] \leqq_{st} G[\sum_1^r \bar{A}_i(Y_i - Y_{i-1})]$$

when $0 \leqq \bar{A}_i \leqq 1$ for $i = 1, 2, \cdots, r$. Choosing $\bar{A}_i = -2 \ln (1 - q)(n - i + 1)/\chi^2_{1-\alpha}(2r)$, we have

$$\begin{aligned} &F[-2 \ln (1 - q)(\chi^2_{1-\alpha}(2r))^{-1}\textstyle\sum_1^r (n - i + 1)(X_i - X_{i-1})] \\ &\qquad \leqq_{st} G[-2 \ln (1 - q)(\chi^2_{1-\alpha}(2r))^{-1}\textstyle\sum_1^r (n - i + 1)(Y_i - Y_{i-1})] \end{aligned}$$

when $-2n \ln (1 - q)/\chi^2_{1-\alpha}(2r) \leqq 1$. It follows that in this case

$$P_F\{1 - F[B_{1-\alpha,q,r}\hat{\theta}_{r,n}] \geqq 1 - q\} \geqq 1 - \alpha.$$

If $-2n \ln (1 - q)/\chi^2_{1-\alpha}(2r) > 1$, then let $\bar{A}_i = (n - i + 1)/n$ so that

$$F[n^{-1}\textstyle\sum_1^r (n - i + 1)(X_i - X_{i-1})] \leqq_{st} G[n^{-1}\sum_1^r (n - i + 1)(Y_i - Y_{i-1})].$$

Also

$$P_G\{1 - G[n^{-1}\textstyle\sum_1^r (n - i + 1)(Y_i - Y_{i-1})] \geqq 1 - q\}$$
$$\geqq P_G\{1 - G[L(\underline{Y})] \geqq 1 - q\} = 1 - \alpha$$

so that (2.1) follows.||

COROLLARY 2.2. *If F is* IFR, $1 - \alpha \geqq 1 - e^{-1}$, *and* $1 - q \geqq e^{-r/n}$, *then*

$$P_F\{1 - F[B_{1-\alpha,q,r}\hat{\theta}_{r,n}] \geqq 1 - q\} \geqq 1 - \alpha.$$

PROOF. By Theorem 2.1 we need only show

$$\chi^2_{1-\alpha}(2r)/2r \geqq -(n/r) \ln (1 - q).$$

Let H denote the chi-square distribution with $2r$ degrees of freedom. Since H is IFR, $H(2r) \leqq 1 - e^{-1}$. This implies $\chi^2_{1-\alpha}(2r) \geqq 2r$, i.e., $\chi^2_{1-\alpha}(2r)/2r \geqq 1$, when $1 - \alpha > 1 - e^{-1}$. Since $1 - q \geqq e^{-r/n}$, then $-(n/r) \ln (1 - q) \leqq 1$. The result follows.||

Theorem 2.1 can be partially extended to IFRA distributions by using inequalities proved in [3].

THEOREM 2.3. *If F is* IFRA, $F(0) = 0$ *and* $F(\xi_q) = q$, *then*

$$P_F\{1 - F[C_{1-\alpha,q,1}X_1] \geqq 1 - q\} \geqq 1 - \alpha,$$

or equivalently,

$$P_F\{\xi_q \geqq C_{1-\alpha,q,1}X_1\} \geqq 1 - \alpha.$$

PROOF. Use Theorem 3.2 of [3] as in the proof of Theorem 2.1 above.||

THEOREM 2.4. *If F is* IFR *and* $\theta = \int_0^\infty x\, dF(x)$, *then*

$$P_F\{\theta \geqq [1 - \exp(-\chi^2_{1-\alpha}(2r)/2n)][2r/\chi^2_{1-\alpha}(2r)]\hat{\theta}_{r,n}\} \geqq 1 - \alpha.$$

PROOF. We use the bound

$$\begin{aligned} F(t;\theta) \geqq b(t;\theta) &= 0, & t < \theta \\ &= 1 - e^{-wt}, & t \geqq \theta \end{aligned}$$

where w depends on t and satisfies $\int_0^t e^{-wx}\, dx = \theta$ (see [2], p. 28). By Theorem 4.2 of [3],

$$G[\textstyle\sum_1^r \bar{A}_i(Y_i - Y_{i-1})] \geqq_{\text{st}} F[\sum_1^r \bar{A}_i(X_i - X_{i-1}); \theta] \geqq_{\text{st}} b[\sum_1^r \bar{A}_i(X_i - X_{i-1}); \theta]$$

if $0 \leqq \bar{A}_i \leqq 1$ for $i = 1, 2, \cdots, r$. Choose $k_{1-\alpha}$ so that

$$P_G\{G[\textstyle\sum_1^r \bar{A}_i(Y_i - Y_{i-1})] \leqq k_{1-\alpha}\} = 1 - \alpha. \tag{2.3}$$

Then

$$P_F\{b[\textstyle\sum_1^r \bar{A}_i(X_i - X_{i-1}); \theta] \leqq k_{1-\alpha}\} \geqq 1 - \alpha.$$

Hence since for $t > \theta$, $b(t;\theta) = 1 - e^{-wt}$ (where $w(\theta)$ satisfies $(1 - e^{-wt})/w = \theta$,

$$P_F\{w(\theta) \leqq -\ln (1 - k_{1-\alpha})/\textstyle\sum_1^r \bar{A}_i(X_i - X_{i-1})\} \geqq 1 - \alpha.$$

Since $w(\theta)$ is decreasing in θ, using the condition just above governing $w(\theta)$, we find

$$P_F\{\theta \geqq -k_{1-\alpha}\textstyle\sum_1^r \bar{A}_i(X_i - X_{i-1})/\ln(1 - k_{1-\alpha})\} \geqq 1 - \alpha.$$

Now choose $\bar{A}_i = c(n - i + 1)$ where $0 \leqq c \leqq 1/n$. Hence (2.3) becomes

$$P_G\{1 - \exp[-c\textstyle\sum_1^r (n - i + 1)(Y_i - Y_{i-1})] \leqq k_{1-\alpha}\} = 1 - \alpha,$$

which implies

$$-2c^{-1}\ln(1 - k_{1-\alpha}) = \chi^2_{1-\alpha}(2r),$$

or

$$k_{1-\alpha} = 1 - \exp[-c\chi^2_{1-\alpha}(2r)/2].$$

Therefore,

$$P_F\{\theta \geqq [1 - \exp(-c\chi^2_{1-\alpha}(2r)/2)]2r\hat{\theta}_{r,n}/\chi^2_{1-\alpha}(2r)\} \geqq 1 - \alpha.$$

To maximize the bound subject to $c \leqq 1/n$, set $c = 1/n$.||

3. Upper confidence limits. It will be convenient to let

$$C^*_{\alpha,q,r} = \max(B_{\alpha,q,r}, r(n - r + 1)^{-1}).$$

THEOREM 3.1. *If F is* IFRA, $F(0) = 0$, *and* $F(\xi_q) = q$, *then*

$$P_F\{F[C^*_{\alpha,q,r}\hat{\theta}_{r,n}] \geqq q\} \geqq 1 - \alpha, \tag{3.1}$$

or equivalently,

$$P_F\{\xi_q \leqq C^*_{\alpha,q,r}\hat{\theta}_{r,n}\} \geqq 1 - \alpha. \tag{3.2}$$

PROOF. Since (3.1) and (3.2) are equivalent, we need only show (3.1). Let $\bar{A}_i = \sum_{j=i}^r a_j$ as before. By Theorem 3.4 of [3],

$$F[\textstyle\sum_1^r \bar{A}_i(X_i - X_{i-1})] \geqq_{st} G[\sum_1^r \bar{A}_i(Y_i - Y_{i-1})]$$

when $a_i \geqq 0$ and $\bar{A}_i \geqq 1$ for $i = 1, 2, \cdots, r$. Hence

$$F\{-2[\ln(1 - q)/\chi_\alpha^2(2r)]\textstyle\sum_1^r (n - i + 1)(X_i - X_{i-1})\}$$
$$\geqq_{st} G\{-2[\ln(1 - q)/\chi_\alpha^2(2r)]\textstyle\sum_1^r (n - i + 1)(Y_i - Y_{i-1})\}$$

when $-2\ln(1 - q)(n - r + 1)/\chi_\alpha^2(2r) \geqq 1$. It follows that in this case,

$$P_F\{F[B_{\alpha,q,r}\hat{\theta}_{r,n}] \geqq q\} \geqq 1 - \alpha.$$

If $-2(n - r + 1)\ln(1 - q)/\chi_\alpha^2(2r) < 1$, then

$$P_G\{G[(n - r + 1)^{-1}\textstyle\sum_1^r (n - i + 1)(Y_i - Y_{i-1})] \geqq q\}$$
$$\geqq P_G\{G[-2\ln(1 - q)(\chi_\alpha^2(2r))^{-1}\textstyle\sum_1^r (n - i + 1)(Y_i - Y_{i-1})] \geqq q\} = 1 - \alpha$$

and (3.1) follows.||

COROLLARY 3.2. *If F is* IFRA, $1 - \alpha \geqq 1 - e^{-1}$, *and* $q \geqq 1 -$

$\exp\{-r/(n-r+1)\}$, *then*

$$P_F\{F[B_{\alpha,q,r}\hat{\theta}_{r,n}] \geqq q\} \geqq 1-\alpha.$$

PROOF. By Theorem 3.1 we need only show

$$\chi_\alpha^2(2r)/2r \leqq -[(n-r+1)/r] \ln(1-q).$$

Let H denote the chi-square distribution with $2r$ degrees of freedom. Since $\ln H(x)$ is concave, $H(2r) \geqq e^{-1}$ by Jensen's inequality, which implies $\chi_\alpha^2(2r) \leqq 2r$, or $\chi_\alpha^2(2r)/2r \leqq 1$, when $1-\alpha > 1-e^{-1}$. Since $1 \leqq -[(n-r+1)/r]\ln(1-q)$ by hypothesis, the result follows.||

It will be convenient to let

$$c_{\alpha,r} = \max(2r/\chi_\alpha^2(2r), r(n-r+1)^{-1}).$$

THEOREM 3.3. *If F is* IFR *and* $\theta = \int_0^\infty x\,dF(x)$, *then*

$$P_F\{\theta \leqq c_{\alpha,r}\hat{\theta}_{r,n}\} \geqq 1-\alpha.$$

PROOF. We use the bound

$$\begin{aligned} F(t;\theta) \leqq B(t;\theta) &= 1-e^{-t/\theta}, && t<\theta \\ &= 1, && t \geqq \theta. \end{aligned}$$

(See [2], p. 27.) By Theorem 3.4 of [3],

$$G[\textstyle\sum_1^r \bar{A}_i(Y_i - Y_{i-1})] \leqq_{st} F[\sum_1^r \bar{A}_i(X_i - X_{i-1});\theta] \leqq_{st} B[\sum_1^r \bar{A}_i(X_i - X_{i-1});\theta]$$

if $a_i \geqq 0$ and $\bar{A}_i \geqq 1$ for $i = 1, 2, \cdots, r$ where $\bar{A}_i = \sum_{j=i}^r a_j$. Choose k_α so that

$$P_G\{G[\textstyle\sum_1^r \bar{A}_i(Y_i - Y_{i-1})] \geqq k_\alpha\} = 1-\alpha.$$

Now let $\bar{A}_i = c(n-i+1)$ for $i = 1, 2, \cdots, r$ where $c \geqq (n-r+1)^{-1}$. Hence as in the proof of Theorem 2.4

$$-\ln(1-k_\alpha) = c\chi_\alpha^2(2r)/2$$

and

$$1-k_\alpha = \exp[-c\chi_\alpha^2(2r)/2].$$

Case 1. $k_\alpha < 1-e^{-1}$ or $c\chi_\alpha^2(2r)/2 < 1$. Now

$$P_F\{B[c\textstyle\sum_1^r (n-i+1)(X_i - X_{i-1});\theta] > k_\alpha\} > 1-\alpha,$$

which implies

$$P_F\{\theta \leqq c[-\ln(1-k_\alpha)]^{-1}\textstyle\sum_1^r (n-i+1)(X_i - X_{i-1})\} \geqq 1-\alpha$$

or

$$P_F\{\theta \leqq 2r\hat{\theta}_{r,n}/\chi_\alpha^2(2r)\} \geqq 1-\alpha.$$

We now choose c as small as possible subject to $c \geqq (n-r+1)^{-1}$; i.e., choose

$c = (n - r + 1)^{-1}$. We do this so that the exponential upper confidence bound will be valid for as many combinations of α and r as possible.

Case 2. $k_\alpha > 1 - e^{-1}$ or $\chi_\alpha^2(2r) \geqq 2(n - r + 1)$. Now

$$P_F\{B[r(n - r + 1)^{-1}\hat{\theta}_{r,n} ; \theta] \geqq k_\alpha\} \geqq 1 - \alpha,$$

which implies

$$P_F\{\theta \leqq r(n - r + 1)^{-1}\hat{\theta}_{r,n}\} \geqq 1 - \alpha. \|$$

Confidence bounds on θ assuming F IFRA can be similarly derived using the probability bounds in [1].

COROLLARY 3.4. *If* F *is* IFR, $\theta = \int_0^\infty x\,dF(x)$, $1 - \alpha > 1 - e^{-1}$, *and* $r \leqq (n + 1)/2$, *then*

$$P_F\{\theta < 2r\hat{\theta}_{r,n}/\chi_\alpha^2(2r)\} \geqq 1 - \alpha.$$

PROOF. By Theorem 3.3 we need only show $\chi_\alpha^2(2r) \leqq 2(n - r + 1)$. As in the proof of Corollary 3.2, $\chi_\alpha^2(2r)/2r \leqq 1$ when $1 - \alpha > 1 - e^{-1}$. Since $(n - r + 1)/r \geqq 1$ when $r \leqq (n + 1)/2$, the result follows.$\|$

4. Confidence limits for DFR distributions.

Confidence limits for DFR and DFRA distributions can also be obtained using the techniques of the previous sections.

Let

$$C^{**}_{1-\alpha,q,r} = \max\,(B_{1-\alpha,q,r}, r(n - r + 1)^{-1}).$$

THEOREM 4.1. *If* F *is* DFRA, *then*

$$P_F\{1 - F[C^{**}_{1-\alpha,q}(r)\hat{\theta}_{r,n}] \geqq 1 - q\} \geqq 1 - \alpha.$$

PROOF. The proof is similar to the proof of Theorem 3.1 where now $F^{-1}G(x)/x$ is increasing in $x \geqq 0$. Hence

$$G[\textstyle\sum_1^r \bar{A}_i(Y_i - Y_{i-1})] \geqq_{st} F[\sum_1^r \bar{A}_i(X_i - X_{i-1})]$$

when $\bar{A}_i \geqq 1$ for $i = 1, 2, \cdots, r$ by Theorem 3.4 of [3]. Letting $\bar{A}_i = -2(n - i + 1)\ln(1 - q)/\chi^2_{1-\alpha}(2r)$ we see that

$$P_F\{1 - F[-2\ln(1 - q)[\chi^2_{1-\alpha}(2r)]^{-1}\textstyle\sum_1^r (n - i + 1)(X_i - X_{i-1})] \geqq 1 - q\} \geqq 1 - \alpha$$

when $-2(n - r + 1)\ln(1 - q)/\chi^2_{1-\alpha}(2r) \geqq 1$. The remainder of the proof is obvious.$\|$

The upper tolerance limits for DFR distributions are not as useful. Let

$$C^{***}_{\alpha,q}(r) = \min\,(B_{\alpha,q,r}, r/n).$$

THEOREM 4.2. *If* F *is* DFR, *then*

$$P_F\{F[C^{***}_{\alpha,q}(r)\hat{\theta}_{r,n}] \geqq q\} \geqq 1 - \alpha.$$

We omit the proof since it is similar to previous proofs.

Let

$$c^*_{\alpha,r} = 2r/\chi^2_{1-\alpha}(2r), \qquad \text{if} \quad \chi^2_{1-\alpha}(2) \leqq 2(n-r+1)$$
$$= r(n-r+1)^{-1} \exp\{1 - \chi^2_{1-\alpha}(2r)[2(n-r+1)]^{-1}\},$$
$$\text{if} \quad \chi^2_{1-\alpha}(2r) \geqq 2(n-r+1).$$

THEOREM 4.3. *If F is* DFR *and $\theta = \int_0^\infty x\, dF(x) < \infty$, then*

$$P_F\{\theta \geqq c^*_{\alpha,r}\hat{\theta}_{r,n}\} \geqq 1 - \alpha.$$

PROOF. We use the bound [2, p. 31]

$$F(t;\theta) \geqq b(t;\theta) = 1 - e^{-t/\theta}, \qquad t \leqq \theta$$
$$= 1 - \theta e^{-1}t^{-1}, \qquad t \geqq \theta.$$

By Theorem 3.4 of [3] with G and F interchanged,

$$G[\textstyle\sum_1^r \bar{A}_i(Y_i - Y_{i-1})] \geqq_{\text{st}} F[\sum_1^r \bar{A}_i(X_i - X_{i-1});\theta] \geqq_{\text{st}} b[\sum_1^r \bar{A}_i(X_i - X_{i-1});\theta]$$

when $\bar{A}_i \geqq 1$ for $i = 1, 2, \cdots, r$.

Choose $k_{1-\alpha}$ so that

$$P_G\{G[\textstyle\sum_1^r \bar{A}_i(Y_i - Y_{i-1})] \leqq k_{1-\alpha}\} = 1 - \alpha.$$

Let $\bar{A}_i = c(n - i + 1)$ for $i = 1, 2, \cdots, r$ so that $c \geqq (n - r + 1)^{-1}$. Then it follows that

$$-2\ln(1 - k_{1-\alpha}) = c\chi^2_{1-\alpha}(2r)$$

as in the proof of Theorem 2.4.

Case 1. $k_{1-\alpha} < 1 - e^{-1}$ or $\chi^2_{1-\alpha}(2r) \leqq 2c^{-1}$. Now

$$P_F\{1 - \exp[-c\theta^{-1}\textstyle\sum_1^r (n-i+1)(X_i - X_{i-1})] \leqq k_{1-\alpha}\} \geqq 1 - \alpha,$$

which implies

$$P_F\{\theta \geqq 2r\hat{\theta}_{r,n}/\chi^2_{1-\alpha}(2r)\} \geqq 1 - \alpha.$$

We want to choose c as small as possible subject to $c \geqq (n - r + 1)^{-1}$. Hence, let $c = (n - r + 1)^{-1}$.

Case 2. $k_{1-\alpha} \geqq 1 - e^{-1}$ or $\chi^2_{1-\alpha}(2r) \geqq 2(n - r + 1)$. In this case,

$$P_F\{1 - (n-r+1)e^{-1}\theta/\textstyle\sum_1^r (n-i+1)(X_i - X_{i-1}) \leqq k_{1-\alpha}\} \geqq 1 - \alpha,$$

which implies

$$P_F\{\theta > (1 - k_{1-\alpha})er(n-r+1)^{-1}\hat{\theta}_{r,n}\} \geqq 1 - \alpha.$$

The bound is obtained by substituting for $1 - k_{1-\alpha}$.||

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