ON A FACTOR AUTOMORPHISM OF A NORMAL DYNAMICAL SYSTEM

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0. Introduction. In this paper we exhibit a factor of a normal dynamical system which, although possessing a similar spectral structure, is not in general isomorphic to a normal dynamical system. In Section 1 we compute the entropy and canonical system of measures associated with the factor decomposition. In Section 2 we obtain a spectral decomposition for the factor automorphism which resembles very closely the usual spectral decomposition of a normal dynamical system. The results of Section 2 are used in Section 3 to give an example of a dynamical system with countable Lebesgue spectrum, and zero entropy which is mixing of all orders. Such an example according to Rohlin [8], has been found by Girsanov (unpublished).

For the theory of Lebesgue spaces and the associated concepts of measurable partitions, homomorphisms, unitary rings c.f. [7], [9].

A dynamical system $(X, \mathcal{B}, m, T)$ (abbreviated to (X, T)) is a Lebesgue space $(X, \mathcal{B}, m)$ together with an automorphism T of $(X, \mathcal{B}, m)$.

Let U_T be the unitary operator induced by T defined on $L^2(X)$ by

$$U_T f = fT,$$

then U_T is an automorphism of the unitary ring $L^2(X)$. If L is a unitary sub-ring of $L^2(X)$ such that $U_T L = L$ we refer to (L, U_T) as a unitary subsystem of the unitary system $(L^2(X), U_T)$.

$(X', \mathcal{B}', m', T')$ is said to be a factor of $(X, \mathcal{B}, m, T)$ if there is a homomorphism ϕ of $(X, \mathcal{B}, m)$ onto $(X', \mathcal{B}', m')$ such that $\phi T = T'\phi$. In this case U_ϕ defined by

$$U_\phi f' = f'\phi$$

will be a ring isomorphism of $(L^2(X'), U_{T'})$ into a unitary subsystem of $(L^2(X), U_T)$.

Let (X', T') be a factor of (X, T) under the homomorphism ϕ. Let $X_{x'} = \{x : \phi(x) = x'\}$ and $\mathcal{B}_{x'} = \{B \cap X_{x'} : B \varepsilon \mathcal{B}\}$; then for almost all $x' \varepsilon X'$ there exists a normalised measure $m_{x'}$ such that $(X_{x'}, \mathcal{B}_{x'}, m_{x'})$ is a Lebesgue space and for every $B \varepsilon \mathcal{B}$, $m_{x'}(B)$ is a measurable function such that

$$\int_{X'} m_{x'}(B)\, dm' = m(B).$$

The measures $m_{x'}$ are called the canonical system associated with the factor decomposition $\phi^{-1}(x')$ and are unique $[m']$.

One way of defining the canonical system $m_{x'}$ is by the formula

$$\int_{X_{x'}} f_n(y)\, dm_{x'} = E(f_n \mid \phi^{-1}\mathcal{B}'),$$

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where this conditional expectation is evaluated at any member of $\phi^{-1}(x')$ and where $\{f_n(x)\}$ is a countable set whose linear span is dense in $L^2(X)$.

Let $\Omega = \prod_{i=-\infty}^{\infty} R_i$, $R_i = R$ the real line, and let $\mathcal{B}$ be the usual product σ-algebra generated by the Borel subsets of R. A measure p_μ is defined on the measurable space $(\Omega, \mathcal{B})$ by the requirement that the co-ordinate sequence $\{x_n(\omega)\}$ $(x_n(\omega) = \omega_n$ if $\omega = \{\omega_n\})$ should be a stationary Gaussian process with covariance sequence

$$R_n(\mu) = \int_I e^{2\pi i\lambda n}\, d\mu = \int_\Omega x_{m+n}(\omega)x_m(\omega)\, dp_\mu$$

where μ is a continuous symmetric normalised measure defined on the Borel subsets of $I = (-\frac{1}{2}, \frac{1}{2}]$; c.f. [4].

Let $(\Omega, \mathcal{B}_\mu, m_\mu)$ be the completion of $(\Omega, \mathcal{B}, p_\mu)$; then $(\Omega, \mathcal{B}_\mu, m_\mu)$ is a Lebesgue space and the shift transformation T_μ,

$$T_\mu\{\omega_n\} = \{\omega_{n+1}\}, \qquad x_n(T_\mu\omega) = x_{n+1}(\omega)$$

is an automorphism. (Ω, T_μ) is called a normal dynamical system. We abbreviate U_{T_μ} to U_μ.

Our aim is to define a factor of (Ω, T_μ). We define an equivalence relation $\sim$ on Ω by: $\omega \sim \alpha$ if and only if $x_n(\omega) = x_n(\alpha)$ for all integers n, or $x_n(\omega) = -x_n(\alpha)$ for all integers n. The partition Ω' of Ω into equivalence classes is a T_μ-invariant partition. The canonical map ϕ which maps each point ω onto the equivalence class ω' which contains it is exactly two to one. The σ-algebra $\mathcal{B}'$ of subsets B' for which $\phi^{-1}(B') \in \mathcal{B}$ has the property that $\phi^{-1}\mathcal{B}'$ is the smallest σ-algebra for which all functions of the form:

$$\prod_{i=1}^{2m} x_{n_i}(\omega) \tag{0.1}$$

are measurable.

Let m' be the measure defined by

$$m'(B') = m(\phi^{-1}B')$$

and let T_μ' be the shift transformation on Ω', i.e., $\phi T_\mu = T_\mu'\phi$. We shall denote the induced unitary operator on $L^2(\Omega')$ by $U_\mu' = U_{T_\mu'}$. It is clear that ϕ induces a ring isomorphism U_ϕ of $L^2(\Omega')$ into $L^2(\Omega)$ such that $U_\phi U_\mu' = U_\mu U_\phi$ whose range $\bar{R}_2$ is the closure of the complex linear span R_2 of functions of the form (0.1). In other words we have

THEOREM 1. *The unitary systems* $(L^2(\Omega'), U_\mu')$, $(\bar{R}_2, U_\mu)$ *are ring isomorphic.*

Let $\mathfrak{X} \subset L^2(\Omega)$ denote the closed complex linear span of $\{x_n(\omega): n = 0, \pm 1, \cdots\}$; $\mathfrak{X}$ is a Gauss linear subspace and if $(y_1, \cdots, y_k) \subset \mathfrak{X}$ are orthogonal then they are independent [3].

The nth Hermite polynomial $H_n(u)$ is defined by

$$H_n(u) = [(-1)^n/n!]e^{u^2/2}(d^n/du^n)e^{-u^2/2}.$$

It is not difficult to show:

(0.2) $H_{2n}(u)$ are polynomials involving even powers only.

$H_{2n+1}(u)$ are polynomials involving odd powers only.

If $\sum_{i=1}^{k} |a_i|^2 = 1$ then there exists constants $a(n_1, \cdots, n_k)$ such that

$$(0.3) \quad H_n(a_1u_1 + \cdots + a_ku_k) = \sum_{n_1+\cdots+n_k=n} a(n_1 \cdots n_k)H_{n_1}(u_1) \cdots H_{n_k}(u_k)$$

([3] p. 106).

Let $Q_n(\mathfrak{X})$ be the linear manifold of functions in $L^2(\Omega)$ of the form

$$f(\omega) = \sum_{n_1+\cdots+n_k=n, n_i \geqq 0} a(n_1 \cdots n_k) \prod_{j=1}^{k} H_{n_j}(y_j(\omega)),$$

where $(y_1(\omega), \cdots, y_k(\omega)) \subset \mathfrak{X}$ is a real orthonormal system and $a(n_1 \cdots n_k)$ are complex constants.

Evidently, $Q_0(\mathfrak{X}) = C$ (the subspace consisting of complex constant functions), $Q_1(\mathfrak{X}) = \mathfrak{X}$,

$$U_\mu Q_n(\mathfrak{X}) = Q_n(\mathfrak{X}), n = 0, 1, \cdots .$$

Moreover, $L^2(\Omega) = \sum_{i=0}^{\infty} \oplus \overline{Q_i(\mathfrak{X})}$ where $\overline{Q_i(\mathfrak{X})}$ is the closure of $Q_i(\mathfrak{X})$ in $L^2(\Omega)$, and the maximal spectral type of U_μ on $\overline{Q_n(\mathfrak{X})}$ is $\mu^n = \mu * \mu^{n-1}$, μ^0 is the normalised measure on I concentrated at 0, and $*$ denotes convolution [2], [3].

1. The factor system $(\Omega', T_{\mu'})$.

LEMMA 1. *If (X', T') is a homomorphic image under ϕ of the ergodic dynamical system (X, T), where $\phi^{-1}(x')$ is countable for almost all $x' \in X'$, then ϕ is essentially n to 1 for some finite cardinal n, in the sense that there exists a partition $\alpha = (A_1, \cdots, A_n)$, of almost all of X, such that $A_i \cap \phi^{-1}(x')$ consists of only one point. Moreover the canonical measures $m_{x'}$ are given by*

$$m_{x'}(A_i \cap \phi^{-1}(x')) = n^{-1}, \qquad i = 1, \cdots, n.$$

PROOF. According to [9] there exists a countable partition $\alpha = (A_1, A_2, \cdots)$, of almost all of X, such that $A_i \cap \phi^{-1}(x')$ consists of only one point, and

$$(1.1) \qquad m_{x'}(A_1 \cap \phi^{-1}(x')) \geqq m_{x'}(A_2 \cap \phi^{-1}(x') \geqq \cdots .$$

It suffices to show that if $m(A_i) > 0$, $m(A_j) > 0$ and $j \geqq i$, then $m(A_i \cap \phi^{-1}(B')) \leqq m(A_j \cap \phi^{-1}(B'))$ for $B' \in \mathfrak{B}'$. Using the ergodicity of T, $A_j \cap \phi^{-1}(B')$ is composed of disjoint parts which are mapped into A_1 under some iterate of T. Without loss of generality we may assume $T^n(A_j \cap \phi^{-1}(B')) \subset A_1$, and $T^n(A_i \cap \phi^{-1}B') \cap A_1 = \varnothing$. Using (1.1) we have

$$m(A_j \cap \phi^{-1}B') = m(T^n(A_j \cap \phi^{-1}B')) \geqq m(T^n(A_i \cap \phi^{-1}B')) = m(A_i \cap \phi^{-1}B').$$

COROLLARY. *Under the hypotheses of Lemma 1 the entropies of T, T' are identical* i.e. $h(T) = h(T')$.

PROOF. We may assume $h(T') < \infty$ for in the contrary case there is nothing to prove. Let ϵ, ϵ' be the partitions of X, X' respectively into individial points. Let β be a partition of X' such that $H(\beta) < \infty$ and $\bigvee_{i=-\infty}^{\infty} (T')^i\beta = \epsilon'$ [10], then $h(T') = h(T', \beta) = H(\beta/\beta^-)$ where $\beta^- = \bigvee_{i=1}^{\infty} (T')^{-i}\beta$. Let α be the finite partition of X defined by Lemma 1. Evidently

$$T(\phi^{-1}\epsilon' \vee \alpha^-) \geqq \phi^{-1}\epsilon' \vee \alpha = \epsilon$$

where $\alpha^- = \bigvee_{i=1}^{\infty} T^{-i}\alpha$, and, consequently, $\phi^{-1}\epsilon' \vee \alpha^- = \epsilon$. Hence for every $\delta > 0$, there exists n such that

$$H(\alpha/T^n\phi^{-1}\beta^- \vee \alpha^-) < \delta.$$

Therefore

$$\begin{aligned} h(T') &= h(T', \beta) = h(T, T^n\phi^{-1}\beta) \leqq h(T) \\ &= h(T, T^n\phi^{-1}\beta \vee \alpha) = H(T^n\phi^{-1}\beta \vee \alpha/T^n\phi^{-1}\beta^- \vee \alpha^-) \\ &\leqq H(\alpha/T^n\phi^{-1}\beta \vee \alpha^-) + H(T^n\phi^{-1}\beta/T^n\phi^{-1}\beta^-) \\ &\leqq \delta + h(T'). \end{aligned}$$

From Lemma 1 and the corollary we deduce:

THEOREM 2. *The canonical measures with respect to the partition Ω' are given by assigning measure $\frac{1}{2}$ to each of the points of $\phi^{-1}(\omega')$, and $h(T_\mu') = h(T_\mu) = 0$ or ∞ according as μ is singular with respect to Lebesgue measure or not.*

PROOF. ϕ is a two to one map, and the entropy of a normal dynamical system is 0 or ∞, according as its spectral measure is singular with respect to Lebesgue measure or not [6].

2. Spectral analysis of (Ω', T_μ'). In view of Theorem 1, an analysis of $(L^2(\Omega'), U_\mu')$ amounts to an analysis of $(\bar{R}_2, U_\mu)$.

LEMMA 2. (i) $Q_{2n}(\mathfrak{X}) \subset \bar{R}_2$, $n = 0, 1, \cdots$;

(ii) $Q_{2n+1}(\mathfrak{X})$ *is orthogonal to* $\bar{R}_2$, $n = 0, 1, \cdots$.

PROOF. (i) It suffices to prove that $\prod_{i=1}^{k} H_{n_i}(y_i(\omega)) \subset \bar{R}_2$ if $n_1 + \cdots + n_k = 2n$, and this follows without much difficulty from (0.2).

(ii) It suffices to show that $\prod_{i=1}^{p} (y_i(\omega))^{n_i}$ is orthogonal to $\prod_{i=1}^{q} H_{m_i}(y_i'(\omega))$ if $\{y_i(\omega), y_i'(\omega)\} \subset \mathfrak{X}$ are real, $y_1'(\omega), \cdots, y_q'(\omega)$ are orthonormal and $n_1 + \cdots + n_p = 2n$, $m_1 + \cdots + m_q = 2m + 1$.

Choose a real orthonormal system $(z_1(\omega), \cdots, z_l(\omega)) \subset \mathfrak{X}$ whose linear span coincidences with the linear span by $(y_1(\omega), \cdots, y_p(\omega), y_1'(\omega), \cdots, y_q'(\omega))$. Then

$$y_i(\omega) = \sum_{j=1}^{l} a_{ij} z_j(\omega), \quad i = 1, \cdots, p, \tag{2.1}$$

and

$$\begin{aligned} y_i'(\omega) &= \sum_{j=1}^{l} a_{ij}' z_j(\omega), \qquad & i = 1, \cdots, q, \\ \sum_{j=1}^{l} |a_{ij}'|^2 &= 1, & i = 1, \cdots, q. \end{aligned} \tag{2.2}$$

Using (2.1), (2.2) and (0.3) the integral

$$\int_\Omega \prod_{i=1}^{p} (y_i(\omega))^{n_i} \prod_{i=1}^{q} H_{m_i}(y_i'(\omega))\, dm_\mu$$

is a linear combination of integrals of the form

$$\int_\Omega \prod_{i=1}^{l} (z_i(\omega))^{n_i'} \prod_{i=1}^{l} H_{m_1{}^i}(z_i(\omega)) \cdots \prod_{i=1}^{l} H_{m_q{}^i}(z_i(\omega))\, dm_\mu \tag{2.3}$$

where $n_1' + \cdots + n_l' = 2n$, $\sum_{i=1}^{l} m_1{}^i + \cdots + \sum_{i=1}^{l} m_q{}^i = 2m + 1$.

The integral (2.3) can be written as

$$\prod_{i=1}^{l} \int_\Omega (z_i(\omega))^{n_i} H_{m_1{}^i}(z_i(\omega)) \cdots H_{m_q{}^i}(z_i(\omega))\, dm_\mu$$

since the orthogonal functions $z_i(\omega)$ are independent.

Since $\sum_{i=1}^{l} (n_i' + m_1^i + \cdots + m_q^i) = 2n + 2m + 1, n_i' + m_1^i + \cdots + m_q^i$ is odd for some i and for this i,

$$\int_\Omega (z_i(\omega))^{n_i'} H_{m_1{}^i}(z_i(\omega)) \cdots H_{m_q{}^i}(z_i(\omega))\, dm_\mu = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} u^{n_i} H_{m_1{}^i}(u) \cdots H_{m_q{}^i}(u) e^{-u^2/2}\, du = 0$$

since the integrand in this latter integral is an odd function. The proof of Lemma 2 is complete.

THEOREM 3. *$\bar{R}_2 = \sum_{n=0}^{\infty} \oplus \overline{Q_{2n}(\mathfrak{X})}$ and the maximal spectral type of U_μ' is $\sum_{n=0}^{\infty} (\mu^{2n}/2^{n+1})$.*

PROOF. The first assertion is immediate from Lemma 2. By Theorem 1 the maximal spectral type of U_μ' is the same as the maximal spectral of U_μ on $\bar{R}_2$ and since the maximal spectral type of U_μ on $\overline{Q_{2n}(\mathfrak{X})}$ is μ^{2n} the theorem is proved. (The numerical factors are introduced to normalise the measure.)

3. An application. We shall need the following lemmas. The properties of convolution of measures that we use may be found in [11]. All the measures referred to in this section are symmetric and defined on the Borel sets of $I = (-\frac{1}{2}, \frac{1}{2}]$.

LEMMA 3. *If $\mu_1 \ll \nu_1$, $\mu_2 \ll \nu_2$ then $\mu_1 * \mu_2 \ll \nu_1 * \nu_2$.*

LEMMA 4. *If $f \varepsilon L^\infty(l)$ where l is Lebesgue measure, and f is symmetric then $f*f$ is bounded, symmetric, continuous and $f*f(0) > 0$ where*

$$f*g(\lambda) = \int_I f(\lambda - t) g(t)\, dl.$$

($\lambda - t$ is taken mod I).

LEMMA 5. *$\mu * l = l$ if $\mu(I) = 1$.*

In the proof of the following lemma $f^n = f*f^{n-1}$.

LEMMA 6. *If $\mu \ll l$ then there exists a positive integer N such that $\mu^N \sim l$.*

PROOF. By Lemma 3 it suffices to prove the lemma for a measure μ such that $\mu(F) = \int_F f(\lambda)\, dl$ where $f \varepsilon L^\infty(l)$. We first note that $\mu^N(F) = \int_F f^N(\lambda)\, dl$. By Lemma 4, $f*f$ is continuous symmetric and $f*f(0) > 0$. It will suffice to prove that, if $\mu(F) = \int_F g(\lambda)\, dl$, where g is non-negative, continuous, symmetric and $g(0) > 0$, then $\mu^N \sim l$ for some positive integer N.

Evidently $g(\lambda) \geqq b\chi_J(\lambda) \equiv h(\lambda)$ for some $b > 0$ and some open neighbourhood of 0, and

$$\mu^N(F) = \int_F g^N(\lambda)\, dl \geqq \int_F h^N(\lambda)\, dl.$$

(The topology of I is obtained by identifying $\frac{1}{2}$ and $-\frac{1}{2}$.)

The support of the continuous function $h^N(\lambda)$ is

$$NJ = \{\lambda_1 + \cdots + \lambda_n \bmod I : \lambda_1, \cdots, \lambda_n \varepsilon J\}$$

and since $\{nJ\}$ is an increasing sequence of open sets whose union is the compact space I, there exists an integer N such that $NJ = I$. Consequently $\mu^N \gg l$ and $\mu^N \ll l$ follows from Lemma 3.

THEOREM 4. *If* $\mu * \mu \ll l$ *then* (Ω', T_μ') *has countable Lebesgue spectrum in the orthogonal complement of the subspace of constant functions.*

PROOF. By Theorem 3 the maximal spectral type of (Ω', T_μ') in the orthogonal complement of the constant function is $l \sim \sum_{n=1}^{\infty} (\mu^{2n}/2^n)$ (Lemmas 3, 5, 6). If $2N$ is the least even integer such that $\mu^{2N} \sim l$ then in each of the orthogonal subspaces $\overline{Q_{2n}(\mathfrak{X})}$, $n = N, N+1, \cdots$, there is a function with Lebesgue spectral type with respect to U_μ. Consequently the multiplicity of the maximal spectral type is $\aleph_0$ and by the separability of $L^2(\Omega')$ this multiplicity is uniform.

EXAMPLE. There is a normalised singular measure μ on I (and consequently there is a symmetric measure μ) such that

$$R_n(\mu) = \int_I e^{2\pi i\lambda n}\, d\mu = O(n^{-\frac{1}{2}+\epsilon}) \quad \text{for every} \quad \epsilon > 0 \qquad ([12] \text{ p. } 146).$$

Since $R_n(\mu^2) = (R_n(\mu))^2$ and $\sum_{n=-\infty}^{\infty} (R_n(\mu))^4 < \infty$ it follows that $\mu^2 \ll l$.

(Ω, T_μ) has zero entropy since μ is singular [6]. (Ω, T_μ) is mixing of all orders since $R_n(\mu) \to 0$ [5].

Consequently (Ω', T_μ') has zero entropy, is mixing of all orders and has countable Lebesgue spectrum (by Theorem 4). Moreover (Ω', T_μ') is not isomorphic to a normal dynamical system since any normal dynamical system with countable Lebesgue spectrum has infinite entropy [6].

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