LIPSCHITZ BEHAVIOR AND INTEGRABILITY OF CHARACTERISTIC FUNCTIONS[1]

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1. I shall establish some theorems connecting the asymptotic behavior of a distribution function F and the local behavior of its characteristic function φ.

Theorem 1. *If* $0 < \gamma < 1$, *we have* $\varphi \varepsilon \operatorname{Lip} \gamma$ *if and only if*

$$F(x) - F(\pm \infty) = O(|x|^{-\gamma}), \qquad |x| \to \infty. \tag{1.1}$$

Condition (1.1) is to be read as $F(x) = O(|x|^{-\gamma})$ as $x \to -\infty$ and $1 - F(x) = O(x^{-\gamma})$ as $x \to +\infty$.

More precisely, $\varphi(x) - \varphi(0) = O(|x|^{\gamma})$, or even $\varphi(x) + \varphi(-x) - 2\varphi(0) = O(|x|^{\gamma})$, implies (1.1); $\varphi \varepsilon \operatorname{Lip} \gamma$ is implied by (1.1). Hence if a characteristic function satisfies a Lipschitz condition of order γ, $0 < \gamma < 1$, at the origin, then it satisfies a Lipschitz condition of the same order at all points.

Theorem 1 fails for $\gamma = 1$. The problem of finding something similar for $\gamma = 1$ is of special interest because it is connected with the problem of the existence of the derivative of a characteristic function at the origin. Let Λ^* and λ^* be the classes of continuous functions φ such that $\varphi(x + h) + \varphi(x - h) - 2\varphi(x) = O(h)$ or $o(h)$, uniformly in x, as $h \to 0$ (λ^* is the class of smooth functions); Λ or λ at x means the same thing for this particular x.

Theorem 2. *We have* $\varphi \varepsilon \Lambda^*$ *or* λ^* *if and only if*

$$F(x) - F(\pm \infty) = O(1/|x|) \quad \text{or} \quad o(1/|x|), \qquad |x| \to \infty. \tag{1.2}$$

More precisely, $\varphi \varepsilon \Lambda$ *or* λ *at* 0 *implies* (1.2); $\varphi \varepsilon \Lambda^*$ *or* λ^* *is implied by* (1.2). *Hence in particular* $\varphi \varepsilon \lambda^*$ *if and only if* φ *is smooth at* 0.

Zygmund [3] showed that $\varphi'(0)$ exists if and only if φ is smooth at 0 and

$$\lim_{T\to\infty} \int_{-T}^{T} t \, dF(t) \quad \text{exists.} \tag{1.3}$$

Pitman [2] showed that $\varphi'(0)$ exists if and only if $F(x) - F(\pm \infty) = o(1/|x|)$ and (1.3) holds. By Theorem 2, we have φ smooth (either at 0 or everywhere) if and only if $F(x) - F(\pm \infty) = o(1/|x|)$, so that the corresponding parts of Zygmund's and Pitman's conditions are really equivalent. In Section 4 I give a short deduction of Pitman's theorem from Theorem 2. (For another proof see Feller [1], p. 528.)

If φ is smooth we can also show that $\varphi'(x)$ exists (for a particular x) if and only if

$$\lim_{T\to\infty} \int_{-T}^{T} te^{ixt} \, dF(t)$$

exists.

Theorems 1 and 2 say that $\varphi(x + h) \to \varphi(x)$ with a specified rapidity if and

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only if $F(t)$ approaches $F(\pm\infty)$ with suitable rapidity. Another way of measuring the rapidity of approach is by the convergence of an integral.

THEOREM 3. *For* $1 < \gamma < 2$, *we have*

$$\int_a^{a+1} (x-a)^{-\gamma} |\varphi(x) - \varphi(a)|\, dx < \infty \tag{1.4}$$

for every real a *if and only if*

$$\int_{-\infty}^{\infty} |t|^{\gamma-1}\, dF(t) \quad \textit{converges.} \tag{1.5}$$

If $\gamma = 1$, $|t|^{\gamma-1}$ *is to be replaced by* $\log^+ |t|$. *If* $\gamma = 2$, i.e. *if the first absolute moment of* dF *is to exist*, (1.4) *is to be replaced by* $\int_0^1 x^{-2} |\varphi(a+x) + \varphi(a-x) - 2\varphi(a)|\, dx < \infty$.

As before, (1.4) with $a = 0$, or even

$$\int_0^1 x^{-\gamma} |\varphi(x) + \varphi(-x) - 2\varphi(0)|\, dx < \infty,$$

implies (1.5); (1.5) implies (1.4) for every a.

2. We begin with Theorem 1, and prove that $\varphi(x) - \varphi(0) = O(|x|^\gamma)$ implies (1.1). Our hypothesis means that

$$\int_{-\infty}^{\infty} (1 - \cos xt)\, dF(t) = O(|x|^\gamma), \qquad x \to 0, \tag{2.1}$$

whence

$$\begin{aligned} \int_0^{1/|x|} (1 - \cos xt)\, dF(t) &= O(|x|^\gamma), \\ \int_0^{1/|x|} x^2t^2(1 - \cos xt)/(x^2t^2)\, dF(t) &= O(|x|^\gamma). \end{aligned} \tag{2.2}$$

Since $(1 - \cos u)/u^2$ decreases on $(0, 1)$, we can replace $(1 - \cos xt)/(x^2t^2)$ by its minimum in (2.2), and get

$$x^2 \int_0^{1/|x|} t^2\, dF(t) = O(|x|^\gamma), \qquad x \to 0,$$

so that

$$A(x) = \int_0^{|x|} t^2\, dF(t) = O(|x|^{2-\gamma}), \qquad |x| \to \infty. \tag{2.3}$$

For $x > 0$,

$$\begin{aligned} 1 - F(x) = \int_x^\infty dF(t) &= \int_x^\infty u^{-2}\, dA(u) \\ &= x^{-2}A(x) + 2\int_x^\infty u^{-3}A(u)\, du \\ &= O(x^{-\gamma}), \qquad x \to +\infty, \end{aligned}$$

by (2.3). For $x \to -\infty$ the argument is similar.

3. To establish the sufficiency of (1.1) take $h > 0$ and write

$$\begin{aligned} \tfrac{1}{2} |\varphi(x+2h) - \varphi(x)| &\leqq \tfrac{1}{2}\int_{-\infty}^{\infty} |e^{2iht} - 1|\, dF(t) \\ &= \int_{-\infty}^{\infty} |\sin ht|\, dF(t) \\ &\leqq h\int_0^{1/h} t\, dF(t) + h\int_{-1/h}^{0} |t|\, dF(t) \\ &\quad + \int_{1/h}^{\infty} dF(t) + \int_{-\infty}^{-1/h} dF(t). \end{aligned}$$

The last two integrals are $O(h^{\gamma})$ by hypothesis. The first integral is

$$\begin{aligned} h\int_0^{1/h} t\,dF(t) &= h\int_0^{1/h} t\,d[F(t) - 1] \\ &= F(1/h) - 1 + h\int_0^{1/h} [1 - F(t)]\,dt \\ &= O(h^{\gamma}); \end{aligned}$$

the second integral is estimated similarly.

4. When $\gamma = 1$ we consider first $\varphi(x) + \varphi(-x) - 2\varphi(0) = O(x)$ or $o(x)$, $x \to 0+$ (i.e., $\varphi \in \Lambda$ at 0 or φ smooth at 0). This means that (2.1) is replaced by

$$\int_{-\infty}^{\infty} (1 - \cos xt)\,dF(t) = O(x) \quad \text{or} \quad o(x), \tag{4.1}$$

whence just as in Section 2 we get

$$\int_0^x t^2\,dF(t) = O(x) \quad \text{or} \quad o(x), \qquad x \to \infty, \tag{4.2}$$

and hence

$$\begin{aligned} 1 - F(x) &= O(1/x) \quad \text{or} \quad o(1/x), && x \to +\infty; \\ F(x) &= O(1/|x|) \quad \text{or} \quad o(1/|x|), && x \to -\infty. \end{aligned} \tag{4.3}$$

Conversely if (4.3) is satisfied then with $h > 0$ we have

$$\begin{aligned} \varphi(x + h) + \varphi(x - h) - 2\varphi(x) &= \int_{-\infty}^{\infty} e^{ixt}[e^{iht} + e^{-iht} - 2]\,dF(t) \\ &= -2\int_{-\infty}^{\infty} e^{ixt}(1 - \cos ht)\,dF(t); \\ |\varphi(x + h) + \varphi(x - h) - 2\varphi(x)| &\leqq 2\int_{-\infty}^{\infty} (1 - \cos ht)\,dF(t) \\ &\leqq h^2\int_0^{1/h} t^2\,dF(t) + h^2\int_{-1/h}^{0} t^2\,dF(t) \\ &\quad + 2\int_{1/h}^{\infty} dF(t) + 2\int_{-\infty}^{-1/h} dF(t). \end{aligned}$$

The last two integrals are $O(h)$ or $o(h)$ by (4.3), and so are the first two if we show that (4.3) implies (4.2). Now if $1 - F(x) = O(1/x)$, $x \to +\infty$, we have

$$\begin{aligned} \int_0^x t^2\,dF(t) &= \int_0^x t^2\,d[F(t) - 1] \\ &= x^2[F(x) - 1] - 2\int_0^x t[F(t) - 1]\,dt \\ &= O(x) + \int_0^x O(1)\,dt = O(x), \end{aligned}$$

and similarly for $x \to -\infty$ and with o replacing O. This establishes Theorem 2.

We next show that if $\varphi'(0)$ exists then

$$\lim_{T\to\infty} \int_{-T}^{T} t\,dF(t) \tag{4.4}$$

exists. In fact, the existence of $\varphi'(0)$ implies in particular that φ is smooth at 0, and hence that $F(x) - F(\pm\infty) = o(1/|x|)$, $|x| \to \infty$. Hence (taking $x > 0$) we have

$$\begin{aligned} \Delta \equiv [\varphi(x) - \varphi(0)]/x - \int_{-1/x}^{1/x} [(e^{ixt} - 1)/xt]t\,dF(t) \\ = x^{-1}\left(\int_{-\infty}^{-1/x} + \int_{1/x}^{\infty}\right)(e^{ixt} - 1)\,dF(t) = o(1). \end{aligned}$$

But for $|xt| < 1$, $(e^{ixt} - 1)/xt = i - R$ with $|R| \leqq Ax\,|t|$, so

(4.5) $\Delta = [\varphi(x) - \varphi(0)]/x - i\int_{-1/x}^{1/x} t\,dF(t) + O(x\int_{-1/x}^{1/x} t^2\,dF(t))$.

Now

$$\int_0^{1/x} t^2\,dF(t) = \int_0^{1/x} t^2\,d[F(t) - 1] = x^{-2}[F(x^{-1}) - 1] - 2\int_0^{1/x} t[F(t) - 1]\,dt$$
$$= o(x^{-1}) + \int_0^{1/x} o(1)\,dt = o(x^{-1}).$$

So the last term in (4.5) is $o(1)$ as $x \to 0$ and the existence of (4.4) follows.

Conversely, if $F(x) - F(\pm\infty) = o(1/|x|)$ and (4.4) exists, the same argument shows that $\Delta \to 0$ and consequently $\varphi'(0)$ exists. This establishes Pitman's result.

A similar argument shows that if $F(x) - F(\pm\infty) = o(1/|x|)$, i.e. if $\varphi \,\varepsilon\, \lambda^*$, then $\varphi'(x)$ exists (for a particular x) if and only if

$$\lim_{T\to\infty} \int_{-T}^{T} e^{ixt}\, t\,dF(t)$$

exists. Also, if $\varphi \,\varepsilon\, \Lambda^*$ then $\varphi \,\varepsilon$ Lip 1 at x if and only if

$$\int_{-T}^{T} e^{ixt}\, t\,dF(t) = O(1);$$

$\varphi \,\varepsilon$ Lip 1 if and only if this is true uniformly in x.

5. We now prove Theorem 3. Suppose first that

$$\int_{-\infty}^{\infty} |t|^{\gamma-1}\,dF(t)$$

converges, $1 < \gamma < 2$, or that $\int_1^{\infty} \log t\,dF(t)$ and $\int_{-\infty}^{-1} \log|t|\,dF(t)$ converge. We have

$$|\varphi(x) - \varphi(a)| = |\int_{-\infty}^{\infty} (e^{ixt} - e^{iat})\,dF(t)|$$
$$\leqq 2\int_{-\infty}^{\infty} |\sin \tfrac{1}{2}(x - a)t|\,dF(t);$$

$$\int_a^{a+1} |x - a|^{-\gamma}\,|\varphi(x) - \varphi(a)|\,dx \leqq 2\int_{-\infty}^{\infty} dF(t)\int_a^{a+1} |x - a|^{-\gamma}\,|\sin \tfrac{1}{2}(x - a)t|\,dx$$
$$= 2\int_{-\infty}^{\infty} dF(t)\int_0^1 u^{-\gamma}\,|\sin \tfrac{1}{2}ut|\,du$$
$$= 2\int_{-\infty}^{\infty} |t|^{\gamma-1}\,dF(t)\int_0^{|t|} s^{-\gamma}\,|\sin \tfrac{1}{2}s|\,ds.$$

If $1 < \gamma < 2$, this converges if

$$\int_{-\infty}^{\infty} |t|^{\gamma-1}\,dF(t)$$

converges, since the inner integral is a bounded function of t. If $\gamma = 1$,

$$\int_0^{|t|} s^{-1}\,|\sin \tfrac{1}{2}s|\,ds \leqq 2 + \log|t|,$$

and the corresponding conclusion follows.

If $\gamma = 2$, we have

$$\varphi(a + x) + \varphi(a - x) - 2\varphi(a) = -2\int e^{iat}(1 - \cos xt)\,dF(t),$$

$$\int_0^1 x^{-2}\,|\varphi(a + x) + \varphi(a - x) - 2\varphi(a)|\,dx \leqq 2\int_{-\infty}^{\infty} dF(t)\int_0^1 x^{-2}(1 - \cos xt)\,dx$$
$$= 2\int_{-\infty}^{\infty} |t|\,dF(t)\int_0^{|t|} u^{-2}(1 - \cos u)\,du.$$

Conversely, suppose that $x^{-\gamma}[\varphi(x) + \varphi(-x) - 2\varphi(0)] \varepsilon L(0, 1), 1 < \gamma \leqq 2$. We have

$$-t^{-\gamma}[\varphi(t) + \varphi(-t) - 2\varphi(0)] = 2t^{-\gamma}\int_{-\infty}^{\infty} (1 - \cos tu)\, dF(u),$$

$$\int_0^1 t^{-\gamma} \left|\varphi(t) + \varphi(-t) - 2\varphi(0)\right| dt = 2\int_{-\infty}^{\infty} dF(u) \int_0^1 t^{-\gamma}(1 - \cos tu)\, dt < \infty.$$

Hence

$$\int_{-\infty}^{\infty} |u|^{\gamma-1}\, dF(u) \int_0^{|u|} s^{-\gamma}(1 - \cos s)\, ds < \infty,$$

and since everything is positive,

$$\int_1^{\infty} u^{\gamma-1}\, dF(u)\int_0^1 s^{-\gamma}(1 - \cos s)\, ds \leqq \int_1^{\infty} u^{\gamma-1}\, dF(u)\int_0^u s^{-\gamma}(1 - \cos s)\, ds < \infty.$$

Consequently $\int_1^{\infty} u^{\gamma-1}\, dF(u) < \infty$. A similar argument applies on the other side.

If $\gamma = 1$, however, we have

$$\int_0^u s^{-1}(1 - \cos s)\, ds \geqq A \log u$$

for large u, and so

$$\int_1^{\infty} \log u\, dF(u) < \infty.$$

Conversely, if $x^{-2} \left|\varphi(x) + \varphi(-x) - 2\varphi(0)\right| \varepsilon L(0, 1)$, we have

$$\int_{-\infty}^{\infty} dF(t)\int_0^1 x^{-2}(1 - \cos tx)\, dx < \infty,$$

$$\int_{-\infty}^{\infty} |t|\, dF(t)\int_0^{|t|} u^{-2}(1 - \cos u)\, du < \infty,$$

$$\int_{|t|>1} |t|\, dF(t)\int_0^1 u^{-2}(1 - \cos u)\, du < \infty,$$

and consequently $\int |t|\, dF(t) < \infty$.

REFERENCES

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