AN OCCUPATION TIME THEOREM FOR THE ANGULAR COMPONENT OF PLANE BROWNIAN MOTION[1]

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1. Summary. Let $Z(t) = (X(t), Y(t))$, $t \geqq 0$, be the standard plane Brownian motion process. Let $(R(t), \theta_1(t))$ be the polar coordinates of $Z(t)$. Suppose that $R(0) = r > 0$ with probability 1, that is $Z(\cdot)$ starts away from the origin. We shall define the process $\theta(t)$, $t \geqq 0$, as the total algebraic angle traveled on the continuous path $Z(s)$, $0 \leqq s \leqq t$; we take $\theta(0) = \theta_1(0)$; we have $\theta(\cdot) \equiv \theta_1(\cdot) \bmod 2\pi$.

Let $f = f(\theta)$, $0 \leqq \theta \leqq 2\pi$, be a bounded measurable function such that $f(0) = f(2\pi)$. For $r_1 > r$ let τ be the first passage time of $R(\cdot)$ to the point r_1. This note is devoted to the computation of the functional

$$L(f) = E_{r,\theta}\{\textstyle\int_0^\tau f(\theta_1(t))\,dt\}, \tag{1.1}$$

where $E_{r,\theta}\{\cdot\cdot\}$ is the expectation operator under the condition $R(0) = r$, $\theta_1(0) = \theta$. This is interpreted as the expected occupation time of a measurable subset of $[0, 2\pi]$ if f is the indicator function of the subset. We find an explicit formula for $L(f)$ as a linear functional on the Hilbert space $L_2[0, 2\pi]$.

A preliminary result of interest is presented in Section 2: the random variable $[\theta(\tau) - \theta(0)]/|\log(r_1/r)|$ has a Cauchy distribution for any positive numbers r, r_1 with $r \neq r_1$. This recalls the independent result of Spitzer that $[\theta(t) - \theta(0)]/\frac{1}{2}\log t$ has a limiting Cauchy distribution for $t \to \infty$ [4].

I thank the referee for his constructive remarks and for the alternate proof of Theorem 2.1 given in Section 6.

2. Distribution of $\theta(\tau)$. The process $\theta(t)$, $t \geqq 0$, has the following representation. Let $U(t)$, $t \geqq 0$, be a standard one-dimensional Brownian motion process independent of $Z(\cdot)$; then,

$$\theta(t) - \theta(0) = U(\textstyle\int_0^t ds/R^2(s)), \qquad t \geqq 0, \tag{2.1}$$

according to Ito and McKean [2], p. 272.

We use this representation in the proof of

Theorem 2.1. *The random variable*

$$[\theta(\tau) - \theta(0)]/|\log(r_1/r)|$$

has the Cauchy distribution for any positive $r \neq r_1$.

Proof. By the representation (2.1) the characteristic function of $\theta(\tau) - \theta(0)$

Received 25 August 1966.

[1] This paper represents results obtained at the Courant Institute of Mathematical Sciences, New York University, under the sponsorship of the National Science Foundation Grant NSF-GP-6237.

is

$$(2.2)\qquad E\{\exp [iuU(\textstyle\int_0^\tau ds/R^2(s))]\} = E\{\exp [(-u^2/2)\textstyle\int_0^\tau ds/R^2(s)]\}.$$

Except for the change of variable $u \to u^2/2$, the latter expression in (2.2) is the Laplace-Stieltjis transform of the distribution function of

$$(2.3)\qquad \textstyle\int_0^\tau ds/R^2(s).$$

We shall find this distribution function and then its transform.

The two-dimensional process

$$(R(t),\ \textstyle\int_0^t ds/R^2(s))$$

is a diffusion process with the following local characteristics: the local mean vector is $(1/2r,\ 1/r^2)$ and the local covariance matrix is the singular matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The distribution of the random variable (2.3) will be derived as a certain absorption probability for the diffusion. For fixed $t > 0$ let the variable s satisfy $0 \leqq s \leqq t$. Let us denote by $p(r, s; r_1, t)$ the probability, under the condition $R(0) = r$, that the sample path $R(\cdot)$ hits r_1 for the first time after the sample path $\int_0^\tau dv/R^2(v)$ hits $t - s$. This probability is the same as

$$(2.4)\qquad P\{\textstyle\int_0^\tau dv/R^2(v) > t - s\}$$

under the condition $R(0) = r$. According to classical diffusion theory [3], p. 48, this absorption probability is the solution of the partial differential equation

$$r^2\partial^2 p/\partial r^2 + r\partial p/\partial r + 2\partial p/\partial s = 0$$

$$p(r_1, s; r_1, t) = 0, \qquad p(r, t; r_1, t) = 1.$$

We solve this by the method of separation of variables: the solution is

$$p(r, s; r_1, t) = (2/\pi) \int_0^\infty e^{-\frac{1}{2}(t-s)y^2} \sin [y|\log (r_1/r)|]\, dy/y.$$

From (2.4) we see that $1 - p(r, 0; r_1, t)$ is the distribution function of the random (2.3); hence, the density function is

$$(1/\pi) \int_0^\infty y e^{-\frac{1}{2}ty^2} \sin [y|\log (r_1/r)|]\, dy.$$

Integration by parts changes this integral to

$$(1/\pi t)|\log (r_1/r)| \int_0^\infty e^{-\frac{1}{2}ty^2} \cos [y|\log (r_1/r)|]\, dy,$$

which, in accordance with the formula

$$(2/\pi)^{-\frac{1}{2}} \int_0^\infty e^{-y^2/2} \cos uy\, dy = e^{-u^2/2},$$

has the value

$$(2\pi t^3)^{-\frac{1}{2}}|\log (r_1/r)| \exp [-|\log (r_1/r)|^2/2t].$$

The Laplace transform with respect to t is well known to be

$$\exp(-|\log(r_1/r)|(2u)^{\frac{1}{2}}), \qquad u \geqq 0;$$

hence, by the remark following equation (2.2), the characteristic function of $\theta(\tau) - \theta(0)$ is $\exp(-|u \log(r_1/r)|)$ which is the characteristic function of the Cauchy distribution with the scale factor $|\log(r_1/r)|$.

We remark that this characteristic function has the form

$$(r/r_1)^{|u|} \tag{2.5}$$

for $r < r_1$.

3. Application of Dynkin's formula. We shall compute $L(f)$ as

$$L(f) = \lim_{\lambda\to 0} E_{r,\theta}\{\textstyle\int_0^\tau e^{-\lambda t} f(\theta(t))\, dt\},$$

where f is a periodic bounded measurable function of θ of period 2π. First we do it for the special case $f(\theta) = e^{im\theta}$ (m = integer) and then extend it to the more general case.

Assume that $r < r_1$; then the expectation operator (1.1) is well defined for a bounded function f because $E\tau < \infty$. We apply Dynkin's fundamental identity for strong Markov processes [1]:

$$E_{r,\theta}\{\textstyle\int_0^\tau e^{-\lambda t + iu\theta(t)}\, dt\} = \phi(\lambda, u; r, \theta) - E_{r,\theta}\{e^{-\lambda\tau}\phi(\lambda, u; r_1, \theta(\tau))\}, \tag{3.1a}$$
$$-\infty < u < \infty, \lambda \geqq 0,$$

where we have put $\phi(\lambda, u; r, \theta) = \int_0^\infty e^{-\lambda t} E_{r,\theta}(e^{iu\theta(t)})\, dt$. The process $Z(\cdot)$ is well known to be stochastically invariant under rotations of the plane around the origin; hence, the conditional joint distribution of the random variables

$$\{\tau, \theta(\tau) - \theta(0), \theta(t) - \theta(0), t \geqq 0\},$$

given $R(0)$ and $\theta(0)$, is independent of $\theta(0)$; therefore, the following relations hold:

$$\phi(\lambda, u; r, \theta) = e^{iu\theta}\phi(\lambda, u; r, 0);$$
$$E_{r,\theta}\{\textstyle\int_0^\tau e^{-\lambda t + iu\theta(t)}\, dt\} = e^{iu\theta} E_{r,0}\{\textstyle\int_0^\tau e^{-\lambda t + iu\theta(t)}\, dt\};$$
$$E_{r,\theta}\{e^{-\lambda\tau}\phi(\lambda, u; r_1, \theta(\tau))\} = e^{iu\theta} E_{r,0}\{e^{-\lambda\tau + iu\theta(\tau)}\}\phi(\lambda, u; r_1, 0).$$

We put $E_r\{\cdot\cdot\} = E_{r,\theta}\{\cdot\cdot\}$ and $\phi(\lambda, u; r) = \phi(\lambda, u; r, 0)$; then equation (3.1a) takes the form

$$E_r\{\textstyle\int_0^\tau e^{-\lambda t + iu\theta(t)}\, dt\} = \phi(\lambda, u; r) - \phi(\lambda, u; r_1) E_{r_1}\{e^{-\lambda\tau + iu\theta(\tau)}\}. \tag{3.1}$$

Ito and McKean [2], p. 271, have computed ϕ:

$$\phi(\lambda, u; r) = K_{|u|}((2\lambda)^{\frac{1}{2}} r) \textstyle\int_0^r I_{|u|}((2\lambda)^{\frac{1}{2}} x) 2x\, dx \tag{3.2}$$
$$+ I_{|u|}((2\lambda)^{\frac{1}{2}} r) \textstyle\int_r^\infty K_{|u|}((2\lambda)^{\frac{1}{2}} x) 2x\, dx,$$

where $K_\nu(z)$ and $I_\nu(z)$ are the Bessel functions as defined in [5], p. 77–78. We substitute the expression on the right hand side of (3.2) in (3.1) and let $\lambda \to 0$; then we find $E_r\{\int_0^T e^{iu\theta(t)}\,dt\}$ by a careful analysis of the right hand side.

4. Asymptotic estimates of the function $\phi(\lambda, u; r)$ for $\lambda \to 0$.

LEMMA 4.1. *For $u \neq 0$,*

$$\lim_{\lambda\to 0} K_{|u|}((2\lambda)^{\frac{1}{2}}r) \int_0^r I_{|u|}((2\lambda)^{\frac{1}{2}}x)2x\,dx = r^2/|u|(|u| + 2).$$

PROOF. Recall the formula [5], p. 183:

$$K_\nu(z) = \tfrac{1}{2}(z/2)^\nu \int_0^\infty \exp\,[-t - z^2/4t]t^{-\nu-1}\,dt. \tag{4.1}$$

It shows that (by the change of variable $t \to z^2/4t$)

$$K_{|u|}((2\lambda)^{\frac{1}{2}}r) \sim 2^{|u|-1}\Gamma(|u|)r^{-|u|}(2\lambda)^{-|u|/2}, \qquad \lambda \to 0,$$

for $|u| \neq 0$. The formula [5], p. 79,

$$I_\nu(z) = [(z/2)^\nu/\Gamma(\nu + \tfrac{1}{2})\Gamma(\tfrac{1}{2})] \int_0^\pi e^{z\cos\theta}\cdot\sin^{2\nu}\theta\,d\theta \tag{4.2}$$

shows that

$$\int_0^r I_{|u|}((2\lambda)^{\frac{1}{2}}x)2x\,dx \sim (2\lambda)^{|u|/2}2^{1-|u|}r^{|u|+2}/(|u| + 2)\Gamma(|u| + 1), \quad \lambda \to 0.$$

The lemma follows from these two estimates.

LEMMA 4.2

$$lim_{\lambda\to 0} I_{|u|}((2\lambda)^{\frac{1}{2}}r) \int_r^\infty K_{|u|}((2\lambda)^{\frac{1}{2}}x)2x\,dx = r^2/|u|(|u| - 2), \quad |u| > 2.$$

PROOF. Formula (4.2) implies that

$$I_{|u|}((2\lambda)^{\frac{1}{2}}r) \sim (2\lambda)^{|u|/2}r^{|u|}/2^{|u|}\Gamma(|u| + 1), \qquad \lambda \to 0.$$

Formula (4.1) implies that

$$\int_r^\infty K_{|u|}(zx)2x\,dx = (2/z^2)\int_0^\infty e^{-t}t^{-(|u|/2)}\,dt \int_{r^2z^2/4t}^\infty y^{|u|/2}e^{-y}\,dy,$$

which, by the change of variable $t \to tz^2$, is equal to

$$2z^{-|u|}\int_0^\infty e^{-tz^2}t^{-(|u|/2)}[\int_{r^2/4t}^\infty y^{|u|/2}e^{-y}\,dy]\,dt$$
$$\sim 2z^{-|u|}\int_0^\infty t^{-(|u|/2)}[\int_{r^2/4t}^\infty y^{|u|/2}e^{-y}\,dy]\,dt, \ (z \to 0).$$

Integration by parts transforms the latter expression into $z^{-|u|}r^{2-|u|}2^{|u|}\Gamma(|u|)/(|u| - 2)$. Putting $z = (2\lambda)^{\frac{1}{2}}$, we obtain the result of the lemma.

LEMMA 4.3. $I_1((2\lambda)^{\frac{1}{2}}r) = r(\lambda/2)^{\frac{1}{2}} + O(\lambda^{\frac{3}{2}}), \lambda \to 0.$

PROOF. Put $z = (2\lambda)^{\frac{1}{2}}$, $\nu = 1$ in formula (4.2); then expand the exponential in the integrand and integrate. Note that the coefficient of λ vanishes because of the relation $\int_0^\pi \cos\theta \sin^2\theta\,d\theta = 0$.

LEMMA 4.4.

$$\int_r^\infty K_1((2\lambda)^{\frac{1}{2}}x)2x\,dx = (1/\lambda)[\int_0^\infty K_1(x)x\,dx - r(2\lambda)^{\frac{1}{2}}yK_1(y)],$$

where y is some number satisfying $0 \leq y \leq r(2\lambda)^{\frac{1}{2}}$.

PROOF. By a change of variable, the integral on the left hand side of the above equation is equal to $(1/\lambda) \int_{r(2\lambda)^{1/2}}^{\infty} K_1(x)x\,dx$. Apply the law of the mean to this integral considered as a function of the lower limit of integration.

LEMMA 4.5.

$$I_2((2\lambda)^{1/2}r) = \lambda r^2/4 + O(\lambda^2).$$

PROOF. The calculation is similar to that for Lemma 4.3.

LEMMA 4.6.

$$\int_r^{\infty} K_2((2\lambda)^{1/2}x)2x\,dx = \lambda^{-1}[r(2\lambda)^{1/2}K_1(r(2\lambda)^{1/2}) + 2K_0(r(2\lambda)^{1/2})].$$

PROOF. The expressions on each side of the equation tend to 0 as $r \to \infty$. The validity of the equation then follows from the equality of the derivatives with respect r, implied by the formulae [5], p. 79:

$$K_0'(x) = -K_1(x), \quad -xK_2(x) = xK_1'(x) - K_1(x).$$

LEMMA 4.7.

$$\lim_{\lambda\to 0} \phi(\lambda, u; r) = 2r^2/(|u| + 2)(|u| - 2), \qquad |u| > 2.$$

PROOF. We apply Lemmas 4.1 and 4.2 and formula (3.2)

LEMMA 4.8.

$$\phi(\lambda, 1; r) = -\tfrac{2}{3}r^2 + r(2\lambda)^{-1/2} \int_0^{\infty} K_1(x)x\,dx + o(1), \qquad \lambda \to 0.$$

PROOF. Apply Lemmas 4.1, 4.3, and 4.4, and formula (3.2). We also use the fact, deducible from (4.1), that

$$\lim_{y\to 0} yK_1(y) = 1. \tag{4.3}$$

LEMMA 4.9.

$$\phi(\lambda, 2; r) = 3r^2/8 + (r^2/2)[-\log r - \log (2\lambda)^{1/2} + C] + o(1), \quad \lambda \to 0,$$

where C is a numerical constant independent of r.

PROOF. Recall the formula [5], p. 80,

$$K_0(z) = -\log z + o(z) + C, \qquad z \to 0, \tag{4.4}$$

where C is a constant; Lemma 4.6 and formula (4.3) imply

$$I_2\,((2\lambda)^{1/2}r) \int_r^{\infty} K_2((2\lambda)^{1/2}x)2x\,dx = r^2/4 + (r^2/2)\,K_0(r(2\lambda)^{1/2} + o(1), \qquad \lambda \to 0.$$

The lemma follows from the formulae (4.4) and (3.2), and Lemma 4.1

5. Computation of $L(f)$. We shall now find the explicit form of the functional $L(f)$ given in (1.1). In accordance with the remarks in Section 3, we omit the subscript θ from the expectation operator.

LEMMA 5.1. *The functional* $L(e^{iu\theta})$ *is equal to*

$$\begin{array}{lr} \tfrac{1}{2}(r_1^2 - r^2), & u = 0, \\ \tfrac{2}{3}r(r_1 - r), & |u| = 1, \\ 2r^2 \log (r_1/r), & |u| = 2, \\ [2/(|u| + 2)\,(|u| - 2)]\,[r^2 - (r/r_1)^{|u|} r_1^2\,], & |u| > 2. \end{array}$$

PROOF. The case $|u| > 2$ is a direct result of equation (3.1), Lemma 4.7, and formula (2.5). The cases $|u| = 1, 2$ follow from equation (3.1), Lemmas 4.8 and 4.9, the formula (2.5), and the asymptotic relation for fixed u and r:

$$E_r[e^{-\lambda\tau + iu\theta(\tau)}] = E_r[e^{iu\theta(\tau)}] + O(\lambda), \qquad \lambda \to 0.$$

The expression for $u = 0$ is apparently well known; it can be verified by calculations similar to those for the other cases, or from the infinitesimal generator of the process $R(t)$.

THEOREM. *Let* $f(\theta)$, $0 \leq \theta \leq 2\pi$, *be a bounded measurable function such that* $f(0) = f(2\pi)$. *Let us denote by* $\{f_n\}$ *its sequence of Fourier coefficients*:

$$f_n = (1/2\pi) \int_0^{2\pi} e^{in\theta} f(\theta)\, d\theta, \qquad n = 0, \pm 1, \cdots;$$

put

$$d_n = E_r\{\int_0^\tau e^{in\theta(t)}\, dt\}, \qquad n = 0, \pm 1, \cdots;$$

then,

$$L(f) = \sum_{n=-\infty}^{\infty} f_n\, d_n\,. \tag{5.1}$$

PROOF. The sequence $\{d_n\}$ is square-summable by Lemma 5.1; hence, the functional $M(f) = \sum_{n=-\infty}^{\infty} f_n\, d_n$ is defined for any f in $L_2[0, 2\pi]$, and is (linear and) continuous over $L_2[0, 2\pi]$. Let us denote by $N(f)$ the restriction of the functional (1.1) to the space $C[0, 2\pi]$ of continuous functions f on $[0, 2\pi]$; $N(f)$ is continuous on $C[0, 2\pi]$. The two functionals M and N are identical for functions f of the form $f(\theta) = e^{im\theta}$, (m = integer); by linearity, they are also identical for linear combinations of such functions. The class of linear combinations of complex exponential functions is dense in $C[0, 2\pi]$ (Weierstrass' theorem), and convergence in $C[0, 2\pi]$ implies convergence in $L_2[0, 2\pi]$; therefore, $N(f) = M(f)$ for all f in $C[0, 2\pi]$ by the continuity of these functionals.

We have shown that $M(f) = L(f)$ for all continuous functions f. The indicator function of an interval is the limit of a monotone sequence of continuous functions, where the convergence is both pointwise and in $L_2[0, 2\pi]$; therefore, M and L coincide over indicator functions since they are positive functionals, and since L is continuous on convergent monotone sequences. The equality of M and L for all bounded measurable functions now follows from conventional approximation arguments.

We point out an alternative form of the functional $L(f)^2$. By the Riesz-Fischer theorem there exists a function g in $L_2[0, 2\pi]$ whose Fourier coefficients are the elements of the sequence $\{d_n\}$. The function g is real-valued because d_n is real and $d_n = d_{-n}$, $n = 0, 1, 2, \cdots$. Parseval's theorem implies that

$$L(f) = (1/2\pi) \int_0^{2\pi} f(\theta)g(\theta)d\theta, \qquad f \text{ in } L_2[0, 2\pi].$$

6. Another proof of Theorem 2.1. The referee has presented another proof of Theorem 2.1. The distribution of the random variable considered in that theorem depends on the pair (r, r_1) only through r_1/r because $\{Z(t)\}$ is stochastically equivalent to $\{c^{-\frac{1}{2}}Z(ct)\}$ for any fixed $c > 0$. Define $\hat{r}$ by the equation $\log \hat{r} = \frac{1}{2} \log (r_1 r)$; let $\hat{\theta}$ correspond to $\hat{r}$ as $\theta(\tau)$ does to r_1; then $\hat{\theta} - \theta(0)$ and $\theta(\tau) - \hat{\theta}$ are independent, identically distributed random variables. By a defining property of the Cauchy distribution it suffices to prove that $\theta(\tau) - \theta(0)(= \theta(\tau) - \hat{\theta} + \hat{\theta} - \theta(0))$ is distributed as $2(\hat{\theta} - \theta(0))$. Now $\log Z(t) = \log R(t) + i\theta_1(t)$ has real and imaginary parts which define a diffusion in R^2 which, since $\log z$ is analytic, differs from $Z(t)$ only by a random time change [2], p. 280. The same is true of $2 \log Z(t)$, or more generally, of $c \log Z(t)$, $c \neq 0$. These processes have identical hitting probabilities; hence the assertion about $\theta(\tau) - \theta(0)$ follows.

[2] ADDED IN PROOF: The functional L has a representation as an integral operator whose kernal is the Green's function for the circle; however, I have not been able to deduce (5.1) directly from it. I would like to thank Zbigniew Ciesielski and Walter Rosenkrantz for their informative comments about this representation.

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